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The AMERICAN MATHEMATICAL MONTHLY

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MATHEMATICAL EDUCATION IN THE U.S.S.R.*

B. V. GNEDENKO,† University of Kiev, U.S.S.R.

During my brief stay in the United States in September, 1956, I was able to visit a number of universities, to meet many American colleagues, and to discuss informally various scientific and pedagogical topics. At some universities I delivered lectures; among these, at Stanford University, I delivered an address on the topic that serves as the title of this article. I got the impression that the lecture provoked a definite interest and that American scientists and teachers wished to know more about mathematical life in the Soviet Union. What I have said explains why I was gratified when Professor R. D. James approached me with the proposal‡ that I write this article for the American Mathematical Monthly. I realize that in a brief article it is not possible to deal fully with all questions that might interest the reader. In order that all who are interested in details of one kind or another, or in general considerations dealt with all too briefly by me, shall be in a position to write to me for fuller explanation, I give my home address.§ As far as possible I shall endeavor to give the necessary information by letter, and, in case I am not competent, I shall refer such enquiries to specialist colleagues, who can give more satisfactory answers. I shall be pleased if my article promotes such correspondence, because a lively exchange of opinions offers to both sides great opportunities for the understanding of various aspects of the organization of teaching methods.

1. General aspects. In present day general education as well as in special education, mathematics occupies an important place. Children of pre-school age encounter it in its simplest form. Later, from the first to the last school year, pupils study in succession arithmetic, elementary algebra, geometry, plane trigonometry, and also the elements of analytic geometry and of mathematical analysis (differential and integral calculus). The transition to special schools (pedagogical, technical, agricultural, *etc.*) and also to higher educational institutions means, for the overwhelming majority of young people, a continuation of mathematical education. Moreover, the completion of higher education and graduation from a university with a diploma does not at all mean the end of further acquaintance with mathematics. Many engineers, teachers, economists, and other specialists are compelled from time to time to turn to mathematics for help, and must study on their own those parts necessary for their immediate needs. Frequently, courses of different types and lectures for specialists who already have their diplomas are organized in the universities of the Soviet Union. All the large cities have institutions for improving the qualifications of

* Translated by J. St. Clair-Sobell and W. H. Simons.

† Regular member of the Academy of Sciences of the Ukrainian Soviet Socialist Republic.

‡ I am indebted to Professor I. J. Schoenberg for recommending this to me. Ed.

§ Sverdlova Str. 14, app. 29, Kiev 3, U.S.S.R.

teachers in the secondary schools. In these institutions, teachers can get advice both on scientific and methodological questions; they can work in the offices and laboratories; they can follow the lectures of specialists.

However, contemporary society cannot be satisfied merely with the transmission of knowledge accumulated by mankind in the past. Each day brings new problems for the solution of which it is necessary not only to avail oneself of methods developed long ago but to create new ones. The present day development of physics and technology makes numerous demands, in the first place, on mathematics, which only a mathematician capable of creating new ideas can satisfy. But also, for the purposes of instruction, persons are required who view their subject as a living and developing discipline and not as a conclusively formed and ossified system of knowledge. This essential idea must pervade all instruction from its first stages to its conclusion. This idea must be central in importance not only in the universities but also in the elementary schools. In fact, the person now being taught in school will become an independent member of society only after some years and, after this, it will be necessary for him to continue working for many more years. What demands life will present to him, what mathematics he will have to use, it is impossible to foresee in school, even along the most general lines. Therefore, even in the first school years, it is necessary to develop in children flexibility of the intellect, to inculcate the thought that in the future it will be necessary to add many more forms of knowledge to the scientific foundation that is laid in the school and, among these, mathematical ones.

In order that this idea should prevail in the schools, it is indispensable to educate teachers in a corresponding spirit. For this purpose, there is in the Soviet Union an extensive network of higher educational institutions in which mathematics serves as a fundamental subject of instruction. These educational institutions are naturally divided into two groups: special faculties of the universities (the mechanical-mathematical faculties of the universities of Moscow, Leningrad, Kiev, Lvov, Saratov, and Tomsk; and the physico-mathematical faculties in the other thirty-one universities) and the physico-mathematical faculties of the pedagogical institutes. There are in the Soviet Union approximately two hundred pedagogical institutes. The programs of these institutes are based on a four-year period of instruction (at present they are in the process of changing to a five-year period) and in these a great deal of attention is devoted to training students in pedagogical skills. In this article I shall not dwell on questions of mathematical education in the pedagogical institutes, since a comparatively recent article by me* is devoted to this subject.

Mathematical instruction in the universities is directed to a considerable degree towards the development of research habits in the students. For this reason, great attention is paid to special courses, special seminars, and course

* Über die Ausbildung der Mathematik und Physiklehrer in der Sowjetunion, Mathematik und Physik in der Schule, vol. 2, 1955, pp. 489–497.

reports. However, some time is devoted also to the development of methodical habits. There will be more about this later when we discuss the universities in greater detail. The university program is based on a five-year period of instruction.

At the completion of university training, persons showing ability for scientific research are able to continue their education in a graduate school. Graduate training continues for three years. During this time the student is obliged to pass several examinations and to prepare a satisfactory serious independent research paper which serves as a dissertation for the advanced degree of Candidate of the Physico-Mathematical Sciences.

In the Soviet Union there is another advanced degree, that of Doctor of the Physico-Mathematical Sciences. To receive this degree it is necessary to develop a new theory and to obtain substantial mathematical results.

After acquiring the degree of Candidate, it is possible to claim the status of docent and, after the defense of the doctoral dissertation, that of professor.

Mathematical education in the Soviet Union has made successful advances in the past forty years. It is possible to judge this, if only by external evidence: the increase in the number of mathematical investigations carried out in the country, the increased publication of books on mathematics, the number of young people who choose mathematics as their special subject, and so on. However, we see not a few defects in our work. Such defects are present both in the organization of instruction in the secondary schools and also in university education. For a long time many problems have been the subject of discussion by the general public, and other problems still await wide discussion. In these discussions, both scholars and teachers and also the parents of the students take part.

2. The content of the school course in mathematics. A secondary education in the U.S.S.R, presupposes the completion of the ten-year school; children are accepted into the schools at the age of seven. The general ten-year course of education is, at present, carried out in all cities and in a number of rural communities. It is proposed that, in the next four or five years, the general ten-year program will be instituted everywhere. An overwhelming number of schools have a completely uniform structure and educational program; only a small number have special characteristics. Such are the secondary schools for musically and artistically gifted children, and the schools with English, German, or French as the language of instruction. In these schools, beginning with the first year, together with the teaching of the general program there are additional lessons in painting, music, or a foreign language. In schools with a foreign language of instruction, learning the language begins in the first year and, starting with the fifth year, the teaching of a number of disciplines in the selected foreign language begins. All the differences just mentioned are in no way reflected in the extent of mathematical instruction, since the program in mathematics is identical for all secondary schools.

During the first five years, the pupils study arithmetic and also receive instruction in elementary notions of geometry. The purpose of instruction in arithmetic is obviously one and the same for all countries of the world: to teach pupils to operate rapidly, confidently, and intelligently with integers and fractions; to apply this knowledge to the solution of problems, including those of a practical nature. In addition to the solution of a very large number of problems and examples in class, the learner must, for each lesson, carry out independently certain assignments at home. Besides this, from time to time, the pupil must formulate his own arithmetical problems. Elementary instruction in geometry has as its aim to acquaint the learner with the simplest geometrical concepts; in particular, with the concepts of the line, the plane figure, the geometrical solid, the length of a line, the area of a surface, and the volume of a solid. The pupils gain knowledge of the metric system of units and they perform the simplest measurements. At this time, they draw up plans of the classroom, the surroundings, home, apartment, *etc.*, and also learn the simplest approximate calculations.

Systematic courses in algebra and geometry are begun in the sixth year and continue to the tenth year. The course in algebra, in addition to an introduction to literal symbols and a considerable training with the object of obtaining experience in the performance of algebraic operations, also includes the solution of equations of the first and second degree, elements of the theory of inequalities, an elementary course in the theory of irrational numbers, the concept of variable and limit, progressions and logarithms, complex numbers, permutations and combinations, and the binomial theorem. In the algebra course, the pupils are systematically instructed in the idea of functional dependence. They become acquainted with the idea of a system of coordinates and they construct graphs of the simplest functions point by point. In recent years, in pedagogical circles, a prolonged discussion has been going on about the expediency of introducing the elements of mathematical analysis into the program of the secondary schools. Many teachers and scholars have expressed the opinion that the elements of differential and integral calculus should become a constituent and permanent part of the school curriculum. As a result of the discussions, it has been decided to work out gradually that volume of knowledge which it is expedient to communicate to the pupils.

The role of geometry in general education is beyond doubt; without question, it contributes more to logical thinking than does any other mathematical discipline of the school course and, at the same time, it develops spatial concepts; in addition, geometrical knowledge has great indirect practical importance. The content of the course in geometry does not present any peculiarities in comparison with what is done in other countries. Geometry is completed in the tenth year with the study of regular polyhedra and the simplest solids of revolution.

Instruction in trigonometry begins in the eighth year and continues to the tenth year. The course includes the theory of the solution of triangles, the basic

formulas for the trigonometric functions, reduction formulas, inverse trigonometric functions, and trigonometric equations.

Throughout the whole period of instruction, along with class work, the pupils have a very considerable amount of homework (the independent solution of problems, the recitation of theorems, the preparation of models, *etc.*). Problems in this connection are selected both of a purely instructional type and also of substantial content, among them, problems associated with school subjects in the natural sciences.

In the instruction, it is strongly advised to avoid all elements of formalism and aridity, to obtain vitality, lucidity, and, at the same time, logical rigor and harmony. It is recommended that considerable attention be given to the general cultural importance of mathematics in all ages, to its many connections with life in the past and at the present time, and that emphasis be placed on the idea of the continuous development of mathematics. In an explanatory note to the program of the secondary schools, the necessity is pointed out for talks with pupils during classes about various important events in the history of our science.

In conversation with me, American colleagues have expressed interest in how well we in the Soviet Union have succeeded in inculcating into all pupils a conscious mastery of all the material of elementary mathematics. It seems necessary for me to say a few words about this here. I think that our schools are still far from a satisfactory solution of this important problem. Even now there is still a serious percentage of students to whom the spirit of mathematical thinking remains alien. Frequently, such pupils even learn the contents of the course fairly well, but their knowledge in this connection remains formal, and a comparatively insignificant change in the conditions of the problem lands them in serious difficulties. Besides the completely obvious diversity in the inclinations of the pupils and the presence of pupils for whom mathematics is a heavy burden, a certain number of students fall into such a situation because of various mistakes in methods made by the teacher, arising out of the lack of skill in instilling interest in the discipline being taught. Moreover, I think that there is a certain amount of formalism in exposition in some textbooks as well. Thus, for example, for the first five years a pupil is occupied with arithmetic and during these years an enormous number of problems are solved. The manuals of methodology attempt to classify rigorously the diversified problems constantly encountered into groups and, for each group, to elaborate specific methods of solution. This in itself is not bad, but what is bad is that a proportion of the teachers elevate these methods and the application of the recommended procedures into sacred canons. As a result, the students mechanically learn these rules and their thinking is not prepared for a conscious approach to each problem. The tendency to maintain mathematical purity in the solution of arithmetical problems, in my view, also complicates the understanding. At the same time, the majority of problems of this type are completely elementary and psychologically clear, and can be solved by the reduction to one or more equations

of the first degree. The introduction into arithmetic of elements of an algebraic approach will simplify the learning of material, relieve students from complicated and artificial methods of solution, and make the approach to arithmetical problems more natural. What is curious is that later, in the seventh year, it is necessary to wean the students from the very methods they learned with difficulty. Moreover, purely arithmetic methods of solution are almost never used in actual practice.

Soviet mathematical circles see a number of defects in the school curriculum. Particularly obvious are the defects in present textbooks. In this connection, a number of prominent mathematicians have taken part in revision of old textbooks and the writing of new ones and also in resolving various questions of the school curriculum. For example, the textbook for arithmetic has been subjected to a significant revision by Professor A. Y. Khinchin; a textbook on algebra has been written by P. S. Alexandrov and A. N. Kolmogorov; L. A. Luisternik and A. F. Bermant have written a textbook on trigonometry; V. L. Goncharov, in a number of books, has worked out methodological questions of the development of functional thinking in students of various ages. It is not possible here to touch on all questions connected with the Soviet school textbook literature; a satisfactory clarification of these questions requires a special article. It is beyond doubt that the fundamental role in the pedagogical process is played by the teacher. A teacher who reduces his task to the point that he only communicates to the pupils that sum of knowledge specified in the curriculum and merely teaches the pupils to deal with routine problems, rarely achieves any success. From the teacher is demanded enthusiasm for his subject and the conviction that his subject is one of the most important affairs of the nation. From the teacher it is demanded that he implant in the students a love for mathematics and a conviction of the creative power of his pupils, and that he describe in general outline before their intellectual gaze, the impressive picture of the uninterrupted development of mathematics with its limitless connections with technology, the natural sciences, and other manifestations of human activity.

3. School mathematics circles. Mathematical Olympiads. The love of mathematics, the encouragement of an inclination to read mathematical books, and the expanding of the mathematical horizon beyond the limits of the compulsory program cannot be developed seriously exclusively during class periods. Once an interest in the science has appeared, it is necessary to develop and to stimulate it daily. In the Soviet Union, a number of steps are being taken which help, in a considerable degree, to bring to light mathematical capabilities in the students and to help them consolidate the mathematical interests that arise. Among steps of this kind belong, first of all, the school mathematics clubs, mathematics circles for school pupils at the universities and other institutions of higher learning, and the school mathematical Olympiads. All these are com-

pletely voluntary both for the pupils and for the teachers. Nevertheless, they attract great interest among the pupils and among a large number of school teachers, instructors in the higher educational institutions, and students and graduate students of the universities, who give willingly of their energy and time to this extra work with young mathematical enthusiasts.

The school mathematics clubs do not have established programs but each works in the direction of the interests of teachers and pupils. As a rule, these clubs pay close attention to anniversaries and observe those of outstanding mathematicians by special evening meetings and by issuing mathematical posters. For example, last year the hundredth anniversary of the death of Gauss was observed and, at the beginning of this year, that of the death of N. I. Lobachevski. At the moment, in many schools, papers are being prepared on the life and creative work of Euler in connection with the imminent two hundred-fiftieth anniversary of his birth. It is understood that, in addition to this purely historical work, the students in the clubs solve complicated problems, examine the most simple logical paradoxes, prepare models, and familiarize themselves with some questions that go beyond the limits of the curriculum. To help the school mathematical circles, several pamphlets have been written and published. By way of example, I refer to the following: A. Y. Khinchin, *Three Pearls in the Theory of Numbers*; Y. I. Perelman, *Living Mathematics*; B. V. Gnedenko and A. Y. Khinchin, *Elementary Introduction to the Theory of Probability*; L. A. Luisternik, *Geodesic Lines*, B. L. Kardemski, *Mathematical Know-How*. Unfortunately, these booklets for the pupils are not arranged in a series, but each is issued separately. In addition to these, there is a series of pamphlets under the general title, *Popular Lessons in Mathematics*, in which there is a considerable amount of well-selected material for the work of the school circles.

We ascribe great importance in the development of an interest in mathematics to a program begun in 1934 by Leningrad University which subsequently spread throughout the Soviet Union and, to some degree, even beyond its borders. I have in mind the so-called mathematical Olympiads. The Olympiads consist of three different parts: (1) the work of the mathematical circles attached to universities or to other higher educational institutions; (2) a series of lectures on various mathematical topics given by university professors; (3) a written contest in the solving of problems.

In the mathematical circles attached to the universities, problems of unusual type that develop the mathematical training of the participants are presented. As a rule, university students and graduate students direct these circles and the pupils are eager to visit and to participate actively in them. As a result of many years of work by such circles attached to Moscow University, a series of original mathematical monographs for pupils has appeared under the general title, *Library of the Mathematical Circle*. Eight books of this series have appeared up to the present time, with the following titles:

Selected problems and theorems of elementary mathematics

1. *Part 1. Arithmetic and algebra* (D. O. Shklyarskii, G. M. Adelson-Velskii, N. N. Chentsov, A. M. Yaglom, I. M. Yaglom).
2. *Part 2. Plane geometry* (D. O. Shklyarskii, N. N. Chentsov, I. M. Yaglom).
3. *Part 3. Solid geometry* (D. O. Shklyarskii, N. N. Chentsov, I. M. Yaglom).
4. *Convex figures* (I. M. Yaglom, V. G. Boltyanskii).
5. *Non-elementary problems discussed in an elementary way* (A. M. Yaglom, I. M. Yaglom).
6. *Mathematical discussion* (E. B. Dynkin, V. A. Uspenski).

Geometrical transformations

7. *Motion and transformation analogies* (I. M. Yaglom).
8. *Linear and circular transformations* (I. M. Yaglom).

The school circles generally work once a week in the evenings.

Lectures for the students are given on Sundays once every two weeks or once a month. These lectures attract a large number of listeners. As examples, I quote the titles of several lectures given at different cities: *Mathematical Paradoxes and Sophisms* (Lvov, 1949), *Regular Polyhedra and Euler's Theorem* (Lvov, 1949), *Geometry of Points* (Moscow, 1949), *The Arithmetic of Residues with Certain Applications* (Moscow, 1949), *Why We Study Mathematics* (Stalingrad, 1953), *Random Walks* (Kiev 1954), *What is a Differential Equation?* (Moscow, 1954), ... The titles quoted naturally do not reflect, even to a small degree, the diversity of the topics of the lectures. The lectures are usually given separately for pupils in the seventh and eighth years and for pupils in the ninth and tenth years.

The final part of the Olympiad is a competition in the solution of problems; generally, two rounds are held. Only those participants of the Olympiad who have successfully solved the problems in the first round are admitted to the second round. In order to convey some idea of the interest of students in the Olympiads, I shall call attention to some figures: 800 students took part in Lvov in 1955; 600 to 950 in Kiev every year; 800 in the first round in Moscow in 1948, 1350 in 1953. The Olympiads in other cities have an equally extensive scope. Because of the large number of participants, the competitions are administered separately to each of the seventh, eighth, ninth, and tenth years. Although in Moscow, as was pointed out, 1350 took part in the first round, only 517 were admitted to the second round. The number who successfully completed the second round was 262. Those who particularly distinguished themselves were awarded prizes (sets of mathematical books, valuable gifts). Three students obtained first class, 15 second class, 24 third class, and 69 received certificates of merit. Elegance and originality in the solutions was taken into account in awarding prizes. After each round, an analysis of the problems proposed was made.

Let me add that newspapers and magazines for children and young people publish articles about mathematics and propose to their readers problems that require ingenuity for their solution.

The mathematical Olympiads materially assist in the selection of talented young people for the new generation of Soviet mathematicians. There is already quite a considerable percentage of former members of the school mathematics circles and prize winners of the Olympiads on the staffs of universities. Among those who enthusiastically direct the school mathematics circles are many who were themselves, a few years before, active members of the circles.

4. Sample problems. I think that it is not without interest to quote a number of problems that were proposed for the Olympiad rounds, as well as a number of those that were solved in the mathematics circles in preparation for the competitions. A large number of collections of such problems had been published by local presses—Moscow, Lvov, and other universities, sometimes in the form of thin pamphlets and sometimes even as small mathematical leaflets. Here are some problems presented in preparation for the eleventh Moscow Olympiad:

1. Every man who has lived on earth has made a certain number of handshakes. Prove that the number of men who have made an odd number of handshakes is even.

2. Prove that $m/3+m^2/2+m^3/6$ is an integer for every integer m.

3. One coin out of 80 golden coins is false (lighter). How can you determine the false coin in four weighings on a scale with two pans and without weights?

4. How many times in 24 hours are the hands of a clock at right angles?

5. Prove that, in any triangle, the sum of the lengths of its three medians is less than its perimeter and greater than its semiperimeter.

6. What is the maximum number of parts into which the plane can be divided by means of 15 straight lines?—of 4 circles?

7. Through a given interior point of a circle construct chords (a) of given length, (b) of minimum length.

8. Prove that all numbers in the sequence 100001, 10000100001, 1000010000100001, $\cdots$, are composite.

9. Construct a triangle given the points where its median, an altitude, and a bisector of one of its angles intersect the circumference of its circumscribed circle.

10. What is the maximum number of acute angles that may occur in a convex polygon?

11. Solve the system of equations $x_1+x_2+x_3=6$, $x_2+x_3+x_4=9$, $x_3+x_4+x_5=3$, $x_4+x_5+x_6=-3$, $x_5+x_6+x_7=-9$, $x_6+x_7+x_8=-6$, $x_7+x_8+x_1=-2$, $x_8+x_1+x_2=2$.

12. Solve the equation $(x+a)(x+2a)(x+3a)(x+4a)=b^4$.

13. Prove that the sum $\frac{1}{2}+\frac{1}{3}+\cdots+(1/n)$ cannot be an integer for any integer n.

14. Rationalize the denominator of the fraction $1/(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c})$.

15. Given four points A, B, C, D, in space such that AB is perpendicular to CD and AC is perpendicular to BD, prove that AD is perpendicular to BC.

16. Into how many parts can n planes divide space?

17. Factor the expression $a^{10}+a^5+1$.

18. Prove that, if α and β are acute angles and $\alpha<\beta$, then $(\tan\alpha)/\alpha<(\tan\beta)/\beta$ and $(\sin\alpha)/\alpha>(\sin\beta)/\beta$.

19. Through a point A inside an angle there passes a straight line which forms, with the sides of the angle, a triangle of minimum area. Prove that the segment of this straight line contained between the sides of the angle is bisected by the point A.

20. Prove that if a, b, c, d are positive numbers and the system of inequalities $ax-by<0$, $cx+dy<0$, $x>0$, $y>0$, has a solution, then the inequality $ad-bc<0$ holds. The converse is also true, namely, if the latter inequality holds, then the system of inequalities has a solution.

I shall now state problems that were given at the time of the competitions. The following were given to pupils of the seventh grade at the 16th Moscow Olympiad.

First round

1. Prove that, in a trapezoid, the sum of the angles subtended by the smaller base is greater than that subtended by the larger base.

2. Find the least integer, made up of 1's only, that is divisible by the number $3\cdots3$, consisting of one hundred 3's.

3. Bisect a line segment using a try square (with a try square, it is possible to construct straight lines and erect perpendiculars, but it is not permissible to drop perpendiculars).

4. Prove that, for every positive integer n, the integer $n^2+8n+15$ is not divisible by $n+4$.

Second round

1. Prove that the g.c.d. of the sum of two numbers and their l.c.m. is equal to the g.c.d. of the numbers themselves.

2. A quadrilateral is circumscribed about a circle. Its diagonals intersect at the center of the circle. Prove that the quadrilateral is a rhombus.

3. Eleven geared wheels are located in a plane in such a way that the first is in mesh with the second, the second with the third, and so on. Finally, the eleventh is in mesh with the first. Can the wheels in such a system turn?

4. A thousand points are the vertices of a 1000-angled convex figure inside which are 500 other points, no three of which lie on the same straight line. The given 1000-angled figure is divided into triangles in such a way that the 1500 given points are the vertices of triangles and these triangles have no other vertices. How many triangles are obtained by such a division?

5. Solve the system of equations

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 &= 1, \\ x_1 + 3x_2 + 4x_3 + 4x_4 + 4x_5 &= 2, \\ x_1 + 3x_2 + 5x_3 + 6x_4 + 6x_5 &= 3, \\ x_1 + 3x_2 + 5x_3 + 7x_4 + 8x_5 &= 4, \\ x_1 + 3x_2 + 5x_3 + 7x_4 + 9x_5 &= 5. \end{aligned}$$

The following problems were given to pupils of the 8th grade in Kishinev in 1953.

First round

1. Prove that the bisectors of the angles of a rectangle, when produced, intersect to form a square.

2. Prove that the product of any four consecutive integers when increased by 1 is a perfect square.

3. Factor $bc(b+c)+ca(c-a)-ab(a+b)$.

4. Prove that two triangles are congruent if two medians and an altitude of one are equal to two medians and an altitude of the other.

Second round

1. Prove that the integer $10\overset{[49]}{\cdots}050\overset{[99]}{\cdots}01$ cannot be the cube of an integer.

2. Prove that, if the roots of the equation $x^2+px+q=0$ are real, then the roots of the equation $x^2+px+q+(x+a)(2x+p)=0$ will be real for all real numbers a.

3. Prove that the straight lines joining the feet of the altitudes of a triangle form a new triangle in which these altitudes of the given triangle bisect the angles of the new triangle.

4. On a given straight line, construct a point such that the difference of its distances from two fixed points is a minimum.

At the Kiev Olympiad in 1954, the following problems were given to pupils of the ninth grade.

First round

1. Determine how many minutes after 4 o'clock it will take for the minute hand to overtake the hour hand.

2. Two cyclists left simultaneously, one traveling from A to B and the other, B to A. Each traveled at a constant speed and, on arrival at the destination point, immediately turned around and returned. They met for the first time 40 km. from B. Eight hours after the first meeting, they met for the second time 20 km. from A. Find the distance from A to B and the speed of each cyclist.

3. The lengths of the sides of a triangle are in arithmetic progression. Prove that the radius of the inscribed circle is one third the length of one of the altitudes.

4. A circle is inscribed in a triangle and a square is circumscribed about the triangle. Prove that more than half the perimeter of the square lies inside the triangle.

5. Let S_m denote the sum of the first m terms of an arithmetic progression. Prove that $3(S_{2m}-S_m)=S_{3m}$.

Second round

1. Prove that the planes passing through the edges of a trihedral angle and bisecting the opposite faces intersect in a single straight line.

2. Rationalize the denominator of the fraction $1/(1-\sqrt[n]{a})$.

3. If, from each of two numbers, one half the smaller is subtracted, then what remains of the larger number is three times what remains of the smaller number. How many times greater than the smaller number is the larger number?

4. In a geometric progression, it is given that $a_{m+n}=A$, $a_{m-n}=B$. Find a_m and a_n.

5. A segment of constant length moves so that its end points are on two straight lines that intersect at right angles. Find the locus of the midpoint of the segment.

The following problems were given to pupils of the tenth grade in the mathematical Olympiad at Ordzhonikidze in 1955.

First round

1. Prove that $n^{n-1}-1$ is divisible by $(n-1)^2$ for an arbitrary integer n.

2. For a triangle with sides a, b, c, and semiperimeter ρ, prove that $a^2+b^2+c^2+1/a+1/b+1/c \geqq 4\rho$.

3. Prove the identity $1!1+2!2+\cdots+n!n=(n+1)!-1$.

4. In an isosceles triangle ABC, $AB=BC=b$, $AC=a$, $\angle ABC=20°$. Prove that $a^3+b^3=3ab^2$.

5. Solve the equation $1/(\sin x)-1/(\cos x)=2\sqrt{2}$.

Second round

1. Prove that all numbers of the form $N=1\overset{[k]}{\cdots}12\overset{[k+1]}{\cdots}25$ are perfect squares and determine the form of $\sqrt{N}$.

2. Two containers A and B, each of the same weight, contain different amounts of water. The weight of A including the water is $\frac{4}{5}$ the weight of B including the water. If all the water in B were emptied into A, then the weight of A would be eight times the weight of B. Given that the water in B weighed 500 gm. more than the water in A, determine the weights of the containers and the original amounts of water in each. (Give arithmetical and algebraical solutions.)

3. If, in a triangular pyramid, sides a, b, c, are mutually orthogonal, and the altitude from the vertex to the base is h, prove that $1/h^2=1/a^2+1/b^2+1/c^2$.

4. Prove that the inequality $\operatorname{ctn}\alpha \geqq 1+\operatorname{ctn} 2\alpha$ is valid if $0<2\alpha<180°$.

5. Determine the maximum and minimum values of the function $y=(x^2+1)/(x^2-x+1)$ and draw its graph.

A considerable number of reports of the Olympiads, together with sample problems, has been published in the journal *Uspehi Mat. Nauk*. It should be said, however, that reports about the majority of the Olympiads have not been published anywhere.

5. The training of mathematicians in the universities. The training of mathematicians in the Soviet Union is carried out, as I mentioned, in 36 universities and in nearly 200 pedagogical institutes. The function of the pedagogical institutes is clear from the name itself; it is necessary to observe, however, that from them come not only the teachers for the secondary schools, but also a certain

number of mathematical specialists. The mathematical training in the universities is much more diverse and much broader. It is natural that the universities are also the basic source of supply of research mathematicians for the scientific institutions, engineering designing offices and laboratories, and for institutions of higher learning. Experience of recent years has shown that the large majority of mathematicians coming from the universities is swallowed up by research institutes and by industry, whereas the universities release only an insignificant number of their graduates for the secondary schools. It is, therefore, particularly important that young people having capabilities and a liking for mathematics should come to the universities. The very process of selecting individuals with mathematical capabilities is exceedingly difficult and we have not yet reached the point where we can attract to the physico-mathematical faculties all those with mathematical capabilities. Moreover, not all who do enter these faculties have such abilities. Certainly, the Olympiads give us remarkable help in the selection of mathematically gifted young people. To a certain degree, entrance examinations aid in this, and only students who have completed their school training with a gold medal are excused from these entrance examinations (but even these must pass a preliminary individual interview with the principal professors of the faculty). A special pamphlet *The Mathematics Profession*, by the academician A. N. Kolmogorov serves as a guide to young people in deciding on their vocation. This pamphlet has been issued in tens of thousands of copies. Certainly, in addition to the search for persons of exceptional mathematical ability, it is no less important to educate young people of average ability up to the level of good, enthusiastic mathematicians. It is understandable that this task depends entirely on the program of instruction, on the quality of the teaching staff of the university, and on those traditions that have been fostered in the student body. It is beyond doubt that the spirit of scientific enthusiasm, if present in the professorial staff, is also transferred to the students. It is also beyond doubt that, if the traditions of the student body are such that scientific knowledge and the desire to further science are fundamental, then the new students entering the university will take up these traditions and these traditions will exert a tremendous influence on the scientific development of youth. To a considerable degree all these factors are of a psychological nature and, important though they are, I shall not examine them in more detail here. I shall limit myself here to an account of the typical educational plans for mathematicians in the universities.

The special mathematical training in Soviet universities has already begun in the first year. In addition to mathematical subjects, the students take only foreign languages (English, French, or German), philosophy, political economy and the history of the party, with also a small amount of compulsory physical activity (two hours a week in the first and second years). For students who are interested in sports, music, choreography, theatricals, *etc.*, there are faculty and university organizations in which they have the opportunity to become proficient in these supplementary interests. Every institution of higher learning

has its own choir, orchestra, ballet, *etc.*, whose membership consists of both students and instructors.

The compulsory courses in mathematics and physics for students specializing in mathematics are given in the following table:

subject	number of hours										
	total	lectures	exercises	laboratory	semester						
					1	2	3	4	5	6	7	8	9
1. Analytical geometry	188	102	86	—	6	5	–	–	–	–	–	–	–
2. Mathematical analysis	476	272	204	—	8	8	6	6	–	–	–	–	–
3. Higher algebra	174	102	72	—	4	3	3	–	–	–	–	–	–
4. Differential geometry	102	68	34	—	–	–	3	3	–	–	–	–	–
5. Differential equations	136	68	68	—	–	–	4	4	–	–	–	–	–
6. Physics	340	136	50	154	–	–	–	4	7	5	4	–	–
7. Equations of mathematical physics	100	70	30	—	–	–	–	–	–	4	2	–	–
8. Theory of functions of a complex variable	100	68	32	—	–	–	–	–	2	4	–	–	–
9. Calculus of variations	30	30	—	—	–	–	–	–	–	–	–	3	–
10. Drawing, elements of descriptive geometry	68	18	50	—	2	2	–	–	–	–	–	–	–
11. Theoretical mechanics	240	150	90	—	–	–	4	4	6	–	–	–	–
12. Astronomy	54	36	—	18	–	–	–	–	3	–	–	–	–
13. Foundations of geometry	64	64	—	—	–	–	–	–	–	–	4	–	–
14. Theory of functions of a real variable and functional analysis	86	86	—	—	–	–	–	–	2	4	–	–	–
15. Integral equations	36	36	—	—	–	–	–	–	–	–	2	–	–
16. Theory of probability	54	54	—	—	–	–	–	–	3	–	–	–	–
17. Theory of numbers	30	30	—	—	–	–	–	–	–	–	–	3	–
18. Elementary mathematics	70	36	34	—	–	–	–	–	–	–	4	–	–
19. Electives	328	200	128	—	–	–	–	–	–	4	4	8	6
20. Mathematical practice, technical work on computers	210	—	36	174	–	–	–	–	4	4	4	3	–
21. Practice teaching	60	—	—	60									
22. Pedagogy	64	64	—	—	–	–	–	–	–	4	–	–	–
23. Mathematical methods	70	40	—	30	–	–	–	–	–	–	4	–	–

Practice teaching is carried out in the eighth semester in secondary schools under the direction of instructors experienced in methodology. The practice teaching lasts six weeks, during which time the students do not have classes at the university.

We should make several observations concerning the program given above. First of all, it must be said that at the end of each semester students are required

to pass examinations in the courses taken. In a large number of courses in which there are exercises, they are also required to make reports to an assistant in which they must give evidence of ability to solve problems of a standard pattern.

In the second, third, and fourth years, students are required to write term papers. As a rule these papers are in the nature of a compilation; they are designed chiefly for developing independence. To produce these papers, students are required to read certain monographs and periodicals and present a written report. In the fifth year each student prepares a graduating thesis. Many of these are in the nature of independent research. The topics of the theses are of great diversity and naturally depend on the scientific interests of the supervisors. The graduating theses are publicly defended by the students, and afterwards each student must pass state examinations. The state examination in mathematics includes the fundamental ideas and results of those mathematics courses taken by the student. The last half-year (tenth semester) is spent, in the majority of cases, in preparing for the state examinations and the graduating thesis.

The topics of the special courses and seminars are exceedingly varied. The student may elect those courses in which he is most interested. Specialization begins with the second year: students choose between mathematics and mechanics; in addition, within mathematics (or mechanics) itself they elect, in the third year, a narrow field of mathematics (or mechanics) such as geometry, topology, algebra, theory of probability, theory of numbers, *etc.* Students elect the majority of special courses in the chosen field. At the same time, the instructors recommend that students attend special courses of a general mathematical nature and also some with a natural science content.

In university circles, discussions are constantly taking place about the nature of the mathematical training of the students. Particularly just now, insistent voices are being heard to the effect that we are lecturing to the students too much and, for this reason, leaving too little time for independent thinking on scientific topics. The proper point of view is put forward that the universities must exert maximal pressure to stimulate active scientific interests in the students. In a number of universities attempts are already being made to decrease the students' work load. Furthermore, the opinion is held that the existing curriculum does not contain a course that would bring mathematical theories together on the basis of the modern status of the science. A number of courses in the curriculum, such as the theory of functions of a real variable, the calculus of variations, and integral equations, are presented to the students as separate and unconnected disciplines from the point of view of ideas, whereas they are, in fact, portions of a single mathematical theory. In this connection, it is proposed to give mathematical analysis in the first and second years and to include, in essence, only the results of the development of mathematics in the 18th and 19th centuries, and to complete analysis with one more course. In relation to this, functional analysis will unite and clarify from a single point of view a series of classical divisions of mathematical analysis. This will enable students to under-

stand the processes of contemporary mathematics more thoroughly, to understand mathematics as a single harmonious whole, and, at the same time, to become acquainted with one of the most important trends in contemporary mathematical creativeness. This idea was tried out as an experiment at Moscow University by Professor A. N. Kolmogorov.

The curriculum for students specializing in the fields of mechanics (general mechanics, theory of elasticity, hydro- and aero-mechanics) is somewhat different from the general curriculum for mathematicians stated earlier. This difference begins in the second year, although at that time, the majority of courses in mechanics are taken together with the mathematicians. The basic division begins in the third year when the greatest number of courses in mechanics are taken separately from the mathematicians. The first ten subjects in the mathematics curriculum coincide exactly with that for mechanics in the number of hours, except that the calculus of variations is taken by students in mechanics in the 7th instead of the 8th semester. We therefore state below only those subjects which have a different number of hours from those of the mathematicians, or those subjects that the mathematicians do not take.

subject	number of hours										
	total	lectures	exercises	laboratory	semester						
					1	2	3	4	5	6	7	8	9
11. Theory of probability	50	36	14	—	–	–	–	–	–	–	–	3	–
12. Mathematical practice. Technique of work with computing machines	104	—	104	—	–	–	–	–	4	2	–	–	–
13. Theoretical mechanics	310	210	100	—	–	–	5	3	6	4	–	–	–
14. Strength of materials	108	60	28	20	–	–	–	–	6	-	–	–	–
15. Hydromechanics. Aeromechanics	170	134	—	36	–	–	–	–	–	4	6	–	–
16. Theory of elasticity	68	68	—	—	–	–	–	–	–	2	2	–	–
17. Electives	336	176	160	—	–	–	–	–	–	–	2	10	10
18. Laboratory practice	176	—	—	176	–	–	–	–	4	2	4	–	–

The mechanics students take their practical work in engineering design offices and in factories. Like the mathematicians, the mechanics students must pass state examinations and, in the tenth semester, complete the writing of the graduating thesis.

The course in mathematical practice is deserving of special mention. The purpose of this course is to compel students to work through, independently, problems requiring for their solution the application of results taken from many mathematical disciplines, the performance of numerical calculations, or the understanding of the physical aspect of the problem. Each student is given a few such problems—five or six a year. We shall give examples of such problems.

Sample problems for the 3rd year

1. Construct a table of values of the function $y(x)$ satisfying the differential equation $y''-xy=0$, with initial conditions $y(x_0)=y_0$, $y'(x_0)=y_0'$. Construct the table for eleven values of x in an arithmetic progression, with first term x_0 and difference 0.02. The calculations of the values must be correct to within 0.001 ($x_0=-3.40$; $y_0=0.1721$; $y_0'=-1.3102$).

2. Given the differential equation $y'=a(x-\alpha)(y-\beta)/y$, where a, α, β, are real numbers and $\alpha<\beta$. (In choosing parameters, one should avoid having $a>10$.) It is required to

1° find the coordinates of singular points and to determine their type;

2° construct integral curves originating at the saddle point, either within the segment $\alpha-1\leqq x\leqq\beta+1$, or up to the points where these integral curves intersect the x-axis;

3° restrict the constructed curves to be within a rectangle having its sides parallel to the coordinate axes; the rectangle should have the property that each of the constructed curves has at most one point in common with the sides of the rectangle; construct one integral curve in each of the regions into which the integral curves originating at the saddle point divide the rectangle.

Note. All graphs should be plotted (a) with such accuracy that the angle between the tangent to the plotted curve and the direction of the field is not greater than 45°, and (b) so that all vertices of the broken curve that approximates the actual curve lie on the same integral curve.

3.1. A telephone station receives three types of calls, (a) ordinary, (b) priority, and (c) "flash." Calls of each type are serviced in the order received. Furthermore, calls of higher category are serviced before calls of lower category. The probability of receiving a call of the ith category during the time dt is $\alpha_i dt$, independently of the distribution of previous calls. The probability that a conversation in progress will end during the time dt is $b dt$, independent of how long it has been in progress. The system is stationary, *i.e.*, a steady state has been established. Find the mean waiting time for subscribers in each of the categories.

3.2. In this part, the same conditions apply as in 3.1, but there are s telephone lines. If, at the time when a call is received, all the lines are in use by subscribers of the same or higher category, the call is refused. In the contrary case, he is immediately connected, and, furthermore, one of the conversations of a lower category is interrupted if necessary for this purpose. Compute the probability of the refusal of a call in each of the categories.

3.3. Two stations A and B are connected by k lines. During an interval dt, A receives a new call with probability $\alpha_1 dt$ and B, with probability $\alpha_2 dt$. If all lines are busy, the calls received by B are refused and subscribers calling A must wait their turn. The probability of concluding a conversation during the time

dt is βdt. The system is stationary. Compute the probability that an incoming call to A will be rejected. Compute the mean waiting period for a subscriber calling B.

4. A system may be in one of the states numbered $1, \cdots, r$. If, at time t the system is in state i, then, during an interval dt, the system may change to state j with the probability $\alpha_{ij}dt$ (independently of its behavior previously). Assuming that a particle leaves state 1 initially, compute the probability that in time t it will reach state 2 exactly n times.

The examples stated show, to some extent, the nature of the problems in the mathematical practice course. Other problems require of the student, a knowledge of the theory of functions of a complex variable, the calculus of variations, *etc.*

It should be noted that each student is given a different set of values of the parameters that may enter into the conditions of the problem.

6. The graduate school. The students who, during the university course, show the best ability are recommended by their professors for further mathematical study in the graduate school. The purpose of graduate training is the education of mathematicians completely dedicated to scientific work. The overwhelming majority of creatively active Soviet mathematicians have gone through graduate training. Under the regulations, graduate training is continued for three years, during which time the student is supported by a grant from the state. Each graduate student is assigned to a supervisor who is consulted whenever necessary for advice of a scientific nature. The supervisor discusses with the student the mathematical literature in the form of periodicals or monographs which the student has read and also makes arrangements with the student regarding plans for his scientific training. Since the interests of the graduate students are extremely varied, there are no standard graduate programs. There are only very general directives about what each student must prepare. Each student must also pass two comprehensive examinations and make reports to his supervisor on two narrowly specialized subjects. For example, one of my students, who is specializing in the theory of probability, must pass an examination in functional analysis that includes Chapters 14–21 of *Functional Analysis and Semigroups*, by E. Hille, *Théorie des Distributions*, by Laurent Schwartz, and the theory of Fourier transforms. His second examination consists of subjects in physics—quantum mechanics, and several chapters of statistical physics. His preparation in the theory of probability consists of a study of the literature on the theory of the summation of random variables and of *Mathematical Methods of Statistics* by H. Cramer. A second student of mine also has functional analysis in his first examination (the theory of Fourier transforms being replaced by *Leçons d'analyse fonctionnelle* by F. Riesz and Bela Sz-Nagy). In his second examination he takes differential equations of parabolic type. His special

interests are in the field of Markov processes; this is taken into account in those special subjects that are included in his program.

In addition to the examinations on mathematical subjects, the graduate student is required to pass an examination in one foreign language and in philosophy. A second language is studied on a voluntary basis. Graduate students are not required to follow lecture courses, but actually each student does attend those special university courses that interest him. The fundamental concern of the graduate student is to do independent scientific work in a selected field. Near the end of his training, he must present independent investigations that contain new mathematical results of scientific value.

The dissertation is defended in public in the presence of two official opponents who are specialists in the given field. After a successful defense, the student receives the degree of Candidate of the Physico-Mathematical Sciences. Many students have already prepared serious scientific papers at the beginning of their graduate training. For them the last year of graduate school is in the nature of work at an institute for advanced study.

Naturally, mistakes do occur in the acceptance of candidates for the graduate schools when incompetent persons, or those who cannot find a topic for investigation that interests them and that they can handle, are accepted. Such graduate students, after receiving additional scientific training, finish graduate school without the defense of a thesis and without obtaining the scientific degree.

A large number of persons in engineering design offices, in institutions of higher learning, and in schools, defend their dissertations without preliminary study at a graduate school. Such persons are required to pass the qualifying examinations beforehand.

The requirements which prevail in the Soviet Union for the degree of Candidate correspond approximately to those for the degree of Ph.D.* in the United States.

In order to receive the Soviet degree of Doctor of Physico-Mathematical Sciences, it is necessary to defend an additional dissertation in which there are much higher scientific requirements. Naturally, there are not nearly as many doctoral dissertations defended in the Soviet Union as there are Candidate dissertations.

Graduate work is given in all Soviet universities and also in all scientific research institutes attached to the Academy of Sciences of the U.S.S.R. and to the Academies of Sciences of the Union Republics. Some pedagogical as well as polytechnical institutes with strong scientific resources are also permitted to give graduate work in mathematics.

In conclusion, I should like to reiterate what I have expressed on more than one occasion in this article for various reasons, namely, we consider that our organization of mathematical education is moderately good, but, for just this reason, we think that it needs further improvement.

* The Russian original is "the degree of Doctor of Sciences." This is, presumably, a misunderstanding on the part of the author. Ed.

FINAL ANALYSIS OF VANDERMONDE'S CONVOLUTION

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1. Introduction. Certain generalizations of a combinatorial identity known as Vandermonde's convolution were discussed in [7]. We wish here to extend these results and illuminate the history of these relations, and in passing exhibit the unity of a number of related expansions which have appeared in this MONTHLY.

Especially in classical mathematics, where time tends to obscure the thin connection between diverse concepts, there can be no better motto than the famous one of E. H. Moore (New Haven Colloquium Lectures, 1906): "The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features."

2. General remarks. We shall use a standard notation to show that the relations in question fit into a general pattern. Most recently they have appeared in diverse forms due to the fact that each special case has been dealt with in modern times from a different approach, but with utmost rigor. It is perhaps not too surprising that we may search back into the literature to find more general results which came to light at a time when questions of convergence and continuity were not as well understood as today.

Acknowledgement must be made of the generous assistance given the author by the late Raymond Clare Archibald who was instrumental in various matters pertaining to the vast literature of mathematics.

3. The Hindenburg combinatorial school. Eminent figures in this eighteenth century period were C. F. Hindenburg, Euler, Lambert, Lagrange, Kramp, Eschenbach, and Toepfer. Typical results were obtained by the use of combinatorial formulae. H. A. Rothe [12] used a notion called "Lokalzeichen" (*signis localibus*) to derive relations which the coefficients in series expansions must satisfy. Netto discusses some of this on pages 238–242 of his famous *Lehrbuch* [9]. A typical result follows.

THEOREM. *Let α, p, and q be integers, and suppose z to be some function of x and that the expansion $x^{\alpha} = \sum_{k=0}^{\infty} K_k(\alpha) z^{p\alpha+qk}$ is valid for some interval of convergence. Then the coefficients must satisfy*

$$\sum_{k=0}^{n} K_k(\alpha) K_{n-k}(\gamma) \cdot (p + qk) = \frac{p(\alpha+\gamma) + qn\alpha}{\alpha+\gamma} K_n(\alpha+\gamma),$$

which is, moreover, an identity in α, γ, p, and q.

What happens is that we make use of an expansion valid for integral exponents and derive a polynomial identity true for all real values of the variables. Vandermonde's convolution is a special case. Such formal methods of treatment

were put into the so-called umbral calculus of Lucas by Blissard about a hundred years ago [1]. The Cauchy product of two series is such a convolution theorem.

THEOREM. *Given a valid representation (for integral* α, β) $x^\alpha = \sum_{k=0}^{\infty} C_k(\alpha, \beta)z^k$, *where* $z = f(x)/x^\beta$ *and* $x^\alpha \cdot g(x, \beta) = \sum_{k=0}^{\infty} G_k(\alpha)z^k$, *where* $G_k(\alpha) = \{(\alpha+\beta k)/\alpha\} \times C_k(\alpha, \beta)$, *then the coefficients must satisfy the following identities for all* α, β, γ, p, *and* q *(real numbers)*:

$$(1) \qquad \sum_{k=0}^{n} C_k(\alpha, \beta)C_{n-k}(\gamma, \beta) = C_n(\alpha + \gamma, \beta);$$

$$(2) \qquad \sum_{k=0}^{n} C_k(\alpha, \beta)C_{n-k}(\gamma, \beta)\cdot(p + qk) = \frac{p(\alpha + \gamma) + qn\alpha}{\alpha + \gamma} C_n(\alpha + \gamma, \beta);$$

$$(3) \qquad \sum_{k=0}^{n} G_k(\alpha)G_{n-k}(\gamma) = \sum_{k=0}^{n} G_k(\alpha + \epsilon)G_{n-k}(\gamma - \epsilon).$$

Relations (10), (11), (12), and (17) in [7] are consequences of these results. All that was needed was to establish a valid series expansion. In the case of [7] we have $f(x) = x - 1$.

Another important result due to Rothe (see also [8], p. 187) is the following:

THEOREM. *Let*

$$y^m = \sum_{k=0}^{\infty} a_{j+\beta k}w^{j+\beta k}, \qquad w^\alpha = \sum_{k=0}^{\infty} \{\alpha/(\alpha + \beta k)\}C_{\alpha+\beta k}y^{(m/j)(\alpha+\beta k)}.$$

Then $C_{\alpha+\beta k}$ *is the coefficient of* $w^{-\alpha}$ *in the expansion of the series*

$$\left(\sum_{i=0}^{\infty} a_{j+\beta i}w^{j+\beta i}\right)^{-(\alpha+\beta k)/j}.$$

4. Illustration of Rothe's theorem. We shall derive (10) as stated in [7] by means of Rothe's theorem above. Let $j = m$, $a_{m+\beta k} = \binom{-m}{k}$, $y = z^{1/\beta}$, and $w = (x-1)^{1/\beta}$. Therefore

$$y^m = z^{m/\beta} = \sum_{k=0}^{\infty} \binom{-m}{k} (x - 1)^{(m+\beta k)/\beta} = (x - 1)^{m/\beta}x^{-m},$$

so that also $z = (x-1)/x^\beta$. Substituting in the various expressions we find that

$$w^\alpha = (x - 1)^{\alpha/\beta} = \sum_{k=0}^{\infty} \{\alpha/(\alpha + \beta k)\}C_{\alpha+\beta k}z^{(\alpha+\beta k)/\beta}$$

or

$$(x - 1)^{\alpha/\beta}g^{-\alpha/\beta} = x^\alpha = \sum_{k=0}^{\infty} \{\alpha/(\alpha + \beta k)\}C_{\alpha+\beta k}z^k,$$

and therefore $C_{\alpha+\beta k}$ is the coefficient of $w^{-\alpha}$ in $z^{-(\alpha+\beta k)/\beta}$, which, in turn, is equal to the coefficient of $(x-1)^{-\alpha/\beta}$ in $(x-1)^{-(\alpha/\beta)-k}x^{\alpha+\beta k}$, the coefficient of $x^{-\alpha/\beta}$ in $x^{-(\alpha/\beta)-k}(x+1)^{\alpha+\beta k}$ and the coefficient of $x^{-\alpha/\beta}$ in

$$\sum_{i=0}^{\infty}\binom{\alpha+\beta k}{i}x^{-(\alpha/\beta)-k+i}.$$

But this coefficient is $\binom{\alpha+\beta k}{k}$ and therefore

$$x^{\alpha}=\sum_{k=0}^{\infty}A_k(\alpha,\beta)z^k, \qquad A_k(\alpha,\beta)=\binom{\alpha+\beta k}{k}\frac{\alpha}{\alpha+\beta k},$$

which demonstrates the result.

5. Schläfli's convolution. Working with a Lambert series, the Swiss mathematician Schläfli ([13], p. 38) found that

$$\sum_{k=0}^{n}\binom{n}{k}P_k(p)P_{n-k}(q)=P_n(p+q), \qquad P_n(p)=p\prod_{j=1}^{k-1}[p+(k-j)\alpha+j\beta)].$$

Thus

$$P_k(p)=(\alpha-\beta)^k k!\,\frac{u}{u+\delta k}\binom{u+\delta k}{k},$$

where we have further made the substitutions $p=(\alpha-\beta)u$, $q=(\alpha-\beta)v$, $\delta=\alpha/(\alpha-\beta)$, so that

$$\sum_{k=0}^{n}A_k(u,\delta)A_{n-k}(v,\delta)=A_n(u+v,\delta).$$

Hence Schläfli's result is equivalent to relation (10) in [7].

6. Abel's series. Schläfli ([13], p. 39) showed that a limiting case of the factorial expansion is the so-called Abel series

$$\sum_{k=0}^{n}\binom{n}{k}(\alpha+\beta k)^k\gamma(\gamma+\beta(n-k))^{n-k-1}=(\alpha+\gamma+\beta n)^n. \tag{5}$$

This relation is equivalent to the Abel series given by Bromwich ([2], p. 200, ex. 21), where Lagrange's inversion expansion is to be used. Bromwich also gives as an exercise ([2], p. 200, ex. 22), the expansion in a series of roots of an equation which is identical with the one used by Schläfli, although Schläfli is not mentioned. The above form of the Abel series is found in the solution to a problem posed by A. C. Aitken (this MONTHLY, vol. 59, 1952, pp. 334–6) and, as a special case, in Otto Dunkel's solution to problem 3103 (this MONTHLY, vol. 32, 1925, pp. 434–6). The series is widely known as a generalization of the binomial theorem which results when $\beta=0$.

Abel's identity follows from the general expansion

$$\sum_{k=0}^{\infty} B_k(\alpha, \beta) z^k = e^{\alpha w}, \qquad z = we^{-\beta w}, \qquad |\beta z| < e^{-1}, \tag{6}$$

where

$$B_k(\alpha, \beta) = \frac{(\alpha + \beta k)^k}{k!} \cdot \frac{\alpha}{\alpha + \beta k}, \qquad e^w = x.$$

This series is analogous to relation (7) in [7]. Discussions of its convergence may be found in [2], p. 160, ex. 4 and in M. S. Klamkin's solution to problem 4489 (this MONTHLY, vol. 60, 1953, p. 485).

Thus, from (6) we have

$$\sum_{n=1}^{\infty} \frac{(\alpha + \beta n)^{n-1}}{n!} z^{n-1} = (e^{\alpha w} - 1)/(\alpha z).$$

If we let $\beta = 1$, $w = 1$, $z = e^{-1}$; $\lim_{\alpha \to 0} (e^{\alpha w} - 1)/(\alpha z) = w/z$, then

$$\sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{n}{e}\right)^{n-1} = e,$$

which is the solution to problem 4489.

The analogue to relation (9) in [7] may be shown to be

$$\sum_{k=0}^{\infty} \frac{\alpha + \beta k}{\alpha} B_k(\alpha, \beta) z^k = e^{\alpha w}/(1 - \beta w), \tag{7}$$

and this for the special case $w = 1$ was called Jensen's formula in [5].

Now, in terms of the second theorem in Section 3, $f(x) = \log x$, $x = e^w$, so that we may find (1), (2), and (3) to be valid for the new coefficients $B_n(\alpha, \beta)$ just as they were valid for $A_n(\alpha, \beta)$. In particular, that is, we have

$$\sum_{k=0}^{n} B_k(\alpha, \beta) B_{n-k}(\gamma, \beta)(p + qk) = \frac{p(\alpha + \gamma) + qn\alpha}{\alpha + \gamma} B_n(\alpha + \gamma, \beta), \tag{8}$$

where $B_k(\alpha, \beta)$ has the same meaning as in (6). Thus, although (5) was not very symmetrical, we may put the Abel series in completely symmetrical forms. The non-symmetrical variety is the more widely known. Other applications occur in problem 3071 (this MONTHLY, vol. 31, 1924, p. 206), and in Dunkel's paper [4].

The mathematician Most* was one of several who used Abel's series to solve the following problem: How can we express the function $F(\alpha) = \sum_{k=0}^{n} A_k \alpha^k$ in the form $F(\alpha) = \sum_{k=0}^{n} B_k(\alpha + \beta k)^k$? The solution is easily obtained as a special case of the more general theorem, true for coefficients in the second theorem of Section 3:

* Grunert's Archiv., vol. 48, p. 104; Fortschritte, vol. 1, 1868, p. 83.

THEOREM. *If* $F(\alpha)=\sum_{k=0}^{n} a_k C_k(\alpha, 0)$, *then*

$$F(\alpha) = \sum_{k=0}^{n} b_k C_k(\alpha, \beta), \tag{9}$$

where $b_k=\sum_{j=k}^{n} a_j C_{j-k}(-\beta j, \beta)$.

7. Generalized differential series. Taylor's series may be put in the form

$$f(\alpha) = \sum_{n=0}^{\infty} (\alpha^{n}/n!) D_t^{n} f(t)\Big]_{t=0}, \tag{10}$$

and a more general Abel series ([2], p. 200, ex. 21; [3], (7.1)) is

$$f(\alpha) = \sum_{n=0}^{\infty} \{\alpha(\alpha-n)^{n-1}/n!\} D_t^{n} f(t)\Big]_{t=n}. \tag{11}$$

We may write a more general expansion now which we shall term a *generalized differential series*,

$$f(\alpha) = \sum_{n=0}^{\infty} B_n(\alpha, \beta) D_t^{n} f(t)\Big]_{t=-\beta n}, \tag{12}$$

where $B_n(\alpha, \beta)$ has the same meaning as in (8). This includes Taylor's series ($\beta=0$) and Abel's series ($\beta=-1$).

8. Generalized difference series. The Newton-Gregory difference expansion may be expressed in the form

$$f(\alpha) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \Delta_{t,1}^{n} f(t)\Big]_{t=0}, \tag{13}$$

where

$$\Delta_{t,z}^{n} f(t) = ((-1)^{n}/z^{n}) \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} f(t+iz).$$

What is essentially the Stirling interpolation series ([3], [10], [11]) is given by

$$f(\alpha) = f(0) + \sum_{n=1}^{\infty} \frac{\alpha}{n} \binom{\alpha-1+n/2}{n-1} \Delta_{t,1}^{n} f(t)\Big]_{t=-n/2}. \tag{14}$$

More generally, we may rewrite (6.1) of [3] as

$$f(\alpha) = \sum_{n=0}^{\infty} A_n(\alpha, \beta) \Delta_{t,1}^{n} f(t)\Big]_{t=-\beta n}, \tag{15}$$

where $A_n(\alpha, \beta)=\{\alpha/(\alpha+\beta n)\}\binom{\alpha+\beta n}{n}$. This may be called a *generalized difference series*. It includes the Newton ($\beta=0$), Stirling ($\beta=\frac{1}{2}$), Selberg-Gelfond ($\beta=-1$), and the special series in [7].

Now, since

$$\lim_{z\to 0} \Delta_{t,z}^n f(t) = D_t^n f(t), \qquad \lim_{z\to 0} \binom{w/z}{n} z^n = w^n,$$

we may combine (12) and (15) into the single generalized difference series expansion

$$f(\alpha + y) = \sum_{n=0}^{\infty} \binom{(\alpha + \beta n)/z}{n} \frac{\alpha}{\alpha + \beta n} z^n \Delta_{t,z}^n f(t + y) \Big]_{t=-\beta n}, \tag{16}$$

which therefore includes all the special cases mentioned. R. C. Buck in [3] gives some general information on the convergence properties of such an expansion. We should like to emphasize that not much detailed convergence information can be obtained in the general case, whereas in all the special function cases cited from this MONTHLY, or from other sources, very detailed data is available. It is apparent from the general theorems (1), (2), (3) and those others of Rothe, that one may prove the polynomial identities of the Vandermonde type [7], the identities of the Abel type (Sec. 6), and others all at once in this sense: If we establish a valid functional expansion by means of (16) and can get it in the form required by Rothe's theorems, then the coefficients necessarily satisfy the same type identities. Thus all the special tricks for proving the Abel identity or the Vandermonde convolution can be obviated if we can establish valid expansions first by the generalized difference expansion (16). As another example of special treatment for the Abel series (5) we may note Otto Hölder's paper *Über eine Verallgemeinerung der binomischen Formel*, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Nat. Kl., vol. 88, 1936, pp. 61–66.

Among the interesting treatments of convolution identities without the use of a theorem such as either of the first two we have quoted in this paper, we must note Chung's method for obtaining the summation identity discussed in [7]. He used graphical induction to solve the following problem. Consider a sequence of independent random variables having the same distribution:

$$x_i = \begin{cases} d - 1 \text{ with probability } 1/d, \\ -1 \text{ with probability } (d-1)/d, \end{cases}$$

and let $S_n = \sum_{i=1}^n x_i$. Consider

$$w_k = Pr\{S_{dk} = 0 \text{ and } S_n \neq 0,\ 0 < n < dk\};\ p_k = Pr\{S_{dk} = 0\}.$$

Then it is known ([6], pp. 243–4) that one has the convolution $p_n = \sum_{k=1}^n w_k p_{n-k}$, $p_0 = 1$, and it is readily verified that

$$p_k = (d-1)^{k(d-1)} d^{-kd} \binom{kd}{k}.$$

Now Chung used graphical induction to show that w_k can be evaluated and the

result is his identity for the binomial coefficients. It would seem that by means of the Rothe theorems together with a method of validating (16) one could generalize such results considerably. These convolutions do arise in probability problems, and another set of results worth mentioning here is that occurring in Dunkel's paper [4].

9. Integration methods. The powerful methods of the Lagrange, Bürmann, and Teixeira theorems (see [14], pp. 129–134) for finding inverse functions may be applied also. Thus, for suitable restrictions on the function f and the contour of integration C, we can write

$$C_k(\alpha, \beta) = \frac{1}{2\pi i}\int_C \frac{x^{\alpha+\beta k}}{\{f(x)\}^k}\left\{\frac{f'(x)}{f(x)} - \frac{\beta}{x}\right\}dx, \tag{17}$$

where we already suppose that we have a valid series of the type in the second theorem of this paper. Thus, if we take $f(x)=x-1$ and integrate around a circle at the origin, we obtain the numbers $A_k(\alpha, \beta)$, and if we take $f(x)=\log x$, we can evaluate $B_k(\alpha, \beta)$ again.

Contour integration should reveal further identities for such coefficients as we have studied here.

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GENERALIZATIONS OF PASCAL'S TRIANGLE*

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1. Introduction. In a recent paper, Freund† proved the formula

$$N_m(r, k) = \sum_{j=0}^{m} N_m(r - j, k - 1) \tag{1}$$

where $N_m(r, k)$ is the number of distinct ways in which r indistinguishable objects can be distributed in k cells allowing at most m objects to fall in each cell. He also showed that the results obtained from (1) could be displayed in a generalized Pascal triangle where the property which was generalized was the rule of formation

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}. \tag{2}$$

We wish to generalize a different property associated with the Pascal triangle, show that this approach provides an alternative proof of Freund's formula, and finally establish another identity involving binomial coefficients. The method would seem to have the advantage of directness and might be of use in establishing other identities.

2. Pascal's triangle. The elements of Pascal's triangle

$$\begin{array}{ccccccc} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ . & . & . & . & . & & \end{array}$$

are, of course, the binomial coefficients $\binom{n}{r}$.

If we think of the elements of the $(n+1)$st row as the coefficients of the expansion of

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r},$$

we see, exchanging x and y, that

* This investigation was supported (in part) by a grant from the National Research Council.

† J. E. Freund, Restricted occupancy theory—a generalization of Pascal's triangle, this MONTHLY, vol. 63, 1956, pp. 20–27.

$$\sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \equiv \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

and conclude that $\binom{n}{r} = \binom{n}{n-r}$. The relation (2) follows at once from the fact that $(x+y)^{n+1} = (x+y)(x+y)^n$ where we adopt the convention that $\binom{n}{r} = 0$ if $n < r$ or if $r < 0$. This relation in turn implies that

$$\sum_{x=s}^{n} \binom{x}{s} = \sum_{x=s}^{n} \left\{ \binom{x+1}{s+1} - \binom{x}{s+1} \right\} = \binom{n+1}{s+1},$$

which shows that, if we add down any column of Pascal's triangle, the sum is equal to the entry below and to the right of the term at which we stop. Thus many identities in the binomial coefficients may be expressed in terms of the Pascal triangle and proved with the help of properties of that triangle.

3. A generalization. If we think of the rows of Pascal's triangle as made up of coefficients of polynomials obtained by multiplying $x+y$ by itself, it is clear that many triangles could be constructed by changing the initial polynomial, $x+y$, and the common multiplier. If we replace $x+y$ by $1+x+\cdots+x^m$, the terms of the $(k+1)$st row will be the coefficients of the combinatorial generating function $(1+x+\cdots+x^m)^k$. If we associate one factor with each of the k cells in the occupancy problem, and select x^i from the corresponding factor if i objects are assigned to a given cell ($i = 0, \cdots, m$), it is clear that $N_m(r, k)$ is the coefficient of x^r in the expansion of the above generating function. Equation (1) then follows.

4. A second generalization. Our interest was aroused in such generalizations when, in the course of a statistical problem, it became necessary to prove the identity

$$\sum_{s=1}^{u} \binom{u}{s} \left\{ \binom{s-1}{u-s-1} - \binom{s-1}{u-s} \right\} = (-1)^u. \tag{3}$$

The left hand side of the identity is clearly the coefficient of x^u in

$$\begin{aligned} f(x; u) &= (x-1) \sum_{s=1}^{u} \binom{u}{s} x^s \sum_{i=0}^{\infty} \binom{s-1}{i} x^i \\ &= (x^2 - x) \sum_{s=1}^{u} \binom{u}{s} x^{s-1}(1+x)^{s-1} \\ &= (x^2 - x) \sum_{s=1}^{u} \binom{u}{s} x^s (1+x)^s / (x^2 + x) \\ &= (x^2 - x)\left[(1 + x + x^2)^u - 1\right] / (x^2 + x) \\ &= \sum_{k=0}^{u-1} (x^2 - x)(1 + x + x^2)^k. \end{aligned}$$

This result suggests forming a generalized Pascal's triangle whose second row consists of the coefficients of $0-x+x^2$, succeeding rows being obtained by multiplication by $1+x+x^2$, and recording the coefficients according to increasing powers of x. The first few rows are

$$
\begin{array}{ccccccccc}
1 & & & & & & & & \\
0 & -1 & 1 & & & & & & \\
0 & -1 & 0 & 0 & 1 & & & & \\
0 & -1 & -1 & -1 & 1 & 1 & 1 & & \\
0 & -1 & -2 & -3 & -1 & 1 & 3 & 2 & 1 \\
. & . & . & . & . & . & . & . & .
\end{array}
$$

We wish to prove that the sum of the elements in the $(u+1)$st column, down to and including the element in the $(u+1)$st row, is $(-1)^u$. To do so, we develop some of the properties of this generalized triangle. Let the polynomial corresponding to the $(k+2)$nd row be

$$(x^2 - x)(1 + x + x^2)^k = \sum_{t=1}^{2k+2} c_{k+1,t}x^t.$$

Then

$$(x^2 - x)(1 + x + x^2)^{k+1} = \sum_{t=1}^{2k+4} c_{k+2,t}x^t = \sum_{t=1}^{2k+2} c_{k+1,t}(x^{t+2} + x^{t+1} + x^t)$$

and

$$c_{k+2,t} = c_{k+1,t} + c_{k+1,t-1} + c_{k+1,t-2}, \tag{4}$$

where $c_{k+1,t}=0$ if $t<1$ or $t>2k+2$.

We can obtain $(x^2-x)(1+x+x^2)^k$ from

$$(x^2 - xy)(x^2 + xy + y^2)^k = \sum_{t=1}^{2k+2} c_{k+1,t}x^t y^{2k+2-t}$$

by setting $y=1$. If we exchange x and y in this identity, we obtain

$$(y^2 - xy)(x^2 + xy + y^2)^k = \sum_{t=1}^{2k+2} c_{k+1,t}x^{2k+2-t}y^t.$$

Setting $y=1$ in both identities and multiplying the second identity by $-x$, we obtain

$$(x^2 - x)(1 + x + x^2)^k \equiv \sum_{t=1}^{2k+2} c_{k+1,t}x^t \equiv \sum_{t=1}^{2k+2} (-c_{k+1,t})x^{2k+3-t}.$$

Hence

(5) $$c_{k+1,t} + c_{k+1,2k+3-t} = 0, \qquad t = 1, \cdots, 2k+2.$$

We let

$$f(x; u) \equiv \sum_{t=1}^{2u} d_{u,t}x^t \equiv \sum_{k=0}^{u-1}\sum_{t=1}^{2k+2} c_{k+1,t}x^t.$$

Then

$$d_{u,t} = \sum_{k=(t-2)/2}^{u-1} c_{k+1,t},$$

where, if $(t-2)/2$ is not an integer, k begins with the next largest integer, and we wish to prove that

$$d_{u,u} = \sum_{k=(u-2)/2}^{u-1} c_{k+1,u} = (-1)^u.$$

The proof will be by induction on u. We note that $d_{1,1} = -1$, and $d_{2,2} = 1$. Consider the case where $u = 2v$, $v > 1$. Then

$$d_{2v,2v} = \sum_{k=v-1}^{2v-1} c_{k+1,2v}, \qquad d_{2v-1,2v-1} = \sum_{k=v-1}^{2v-2} c_{k+1,2v-1},$$

and, by (4) and (5),

$$\begin{aligned}
d_{2v,2v} &= c_{2v,2v} + \sum_{k=v-1}^{2v-2} c_{k+1,2v} \\
&= c_{2v-1,2v} + c_{2v-1,2v-1} + c_{2v-1,2v-2} + \sum_{k=v-1}^{2v-1} c_{k+1,2v} \\
&= -c_{2v-1,2v-1} + (c_{2v-1,2v-1} + c_{2v-1,2v}) + c_{2v-1,2v-2} + \sum_{k=v-1}^{2v-3} c_{k+1,2v} \\
&= -c_{2v-1,2v-1} + c_{2v-1,2v-2} + \sum_{k=v-1}^{2v-3} c_{k+1,2v} \\
&= -\sum_{k=v}^{2v-3} c_{k+1,2v-1} + c_{v+1,2} + c_{v,2v} \\
&= -\sum_{k=v}^{2v-2} c_{k+1,2v-1} + c_{v,2} + (c_{v,1} + c_{v,2v}) + c_{v,0} \\
&= -\sum_{k=v}^{2v-2} c_{k+1,2v-1} - c_{v,2v-1} = -d_{2v-1,2v-1}.
\end{aligned}$$

In the same way, we prove $d_{2v+1,2v+1} = -d_{2v,2v}$ and the proof is complete.

RESIDUES OF IDEALS

(after a mathematicians' party)

The room is filled with emptiness and smoke
The rug looks bored and sullen—nostalgic for laced oxfords, probably,
The large, brown, weighty kind that has impressed upon it for an evening
The necessary and sufficient condition of existence.

The chairs, which had been twelve at eight o'clock,
Twelve four-legged, seatable commodities, worth sixteen dollars each,
Are turned into a countable set of points, dense almost everywhere
And scattered random-fashion within party limits.

The tables, freed for five short hours from humdrum horizontalness
(Thanks to mathy minds) had taken to the air as intersecting, real,
Or tangential planes. But they came in now—with a crash (it's 1:AM)
The debris spreading wide. A lemma lost its sense
And lies dead on the floor, covered with paper napkins.
Toothpicks on window sills. Cherry stones. And salad leaves.
And fragmentary sandwich matter. Prime ideals, lower radicals
Smoldering among the ashes of cremated Chesterfields.

And at the bottom of a tall and empty glass a theorem lies quite dead (and wet)
The experts will identify the body.
Someone will bury it beside the lemma in a journal
And hide the grave under a corollary.

ISOTTA CESARI

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Material for this department should be sent to Ivan Niven, Department of Mathematics, University of Oregon, Eugene, Oregon.

OVALS WITH EQUICHORDAL POINTS

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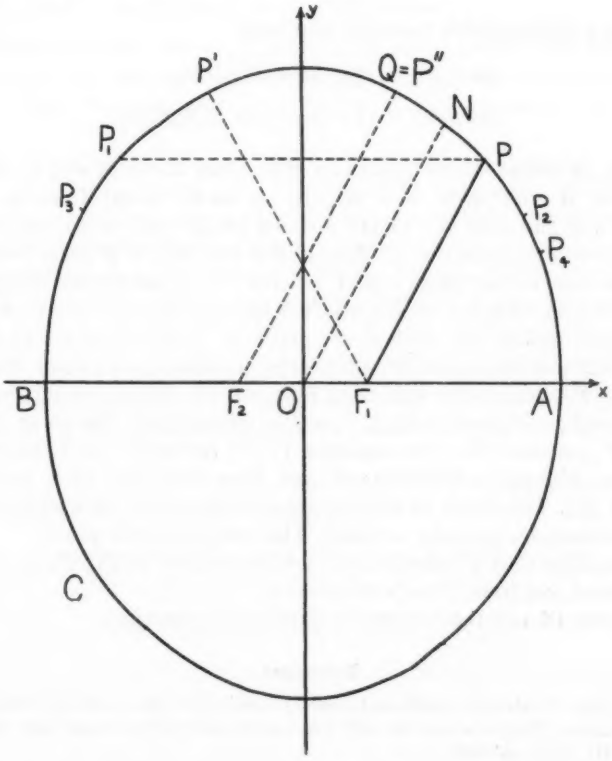
A point F is called an equichordal point with respect to a curve C if all chords of C passing through F have equal length. Dirac* [1] has proved that if C is an oval (a bounded closed convex plane curve of class C^2) it cannot possess

* Dirac attributes this problem to Erdös, which is incorrect (cf. Math. Reviews, vol. 14, 1953, p. 309). The problem seems to be proposed by Blaschke, Rothe, and Weitzenböck (Aufgabe 552, Arch. Math. Phys., vol. 27, 1917, p. 82). It has been treated also by Süss [4], Dulmage [2] and Helfenstein [3].

more than two equichordal points, and if there are two such points F_1 and F_2, say, the oval must have the following properties: (i) C is symmetric about the line (F_1F_2); (ii) C is symmetric about the line orthogonal to (F_1F_2) passing through the point O, the midpoint of $[F_1F_2]$; (iii) if $r=r(\alpha)$ is the equation of C with respect to the pole F_1 (F_2 resp.) and (F_1F_2) as the polar axis (oriented as $[F_2F_1]$), then $r(\alpha)$ is a strictly increasing (resp. decreasing) function of α for $0 \leqq \alpha \leqq \pi$. We will show that an oval possessing two equichordal points and the properties (i)–(iii) must be necessarily a circle. Consequently, no oval can have more than one equichordal point.

Proof. Let C be the oval (see the diagram), F_1 and F_2 the equichordal points, $AB=1$, $OF_1=OF_2=a<1/2$. Let (α, r) be the polar coordinates of the point P with respect to the pole F_1 and the ray $[F_1X)$; we choose $0 \leqq \alpha \leqq \pi$. With every



point P we associate the unique point P' on C whose polar angle is $\pi-\alpha$. Then from the symmetry properties (i) and (ii) it follows $F_1P'=r(\pi-\alpha)=1-r(\alpha)$,

and also $F_1P' = F_2Q$ where F_2Q is parallel to F_1P and Q is a point on C. Thus, if ON is parallel to F_1P, $ON \geqq 1/2$ from the convexity of C, and the equality sign holds for $\alpha = 0$ or $\alpha = \pi$ only (unless $a = 0$), since we note the simple fact that no segment can belong to C.

Let $R = R(\beta)$ be the polar equation of C with respect to the pole O and the ray $[OX)$. Since $R(0) = 1/2$ it follows that $R(\beta)$ is strictly increasing in an interval $[0, \beta_0]$, $\beta_0 > 0$. For each α, $\beta_1(\alpha)$ and $\beta_2(\alpha)$ are the polar angles of P and P' respectively, where P is the unique point of C having angular coordinate α relative to F_1 as pole.

The relations

$$[R(\beta_1(\alpha))]^2 = r(\alpha)^2 + a^2 + 2ar(\alpha)\cos\alpha,$$

$$[R(\beta_2(\alpha))]^2 = (1 - r(\alpha))^2 + a^2 - 2a(1 - r(\alpha))\cos\alpha,$$

define R as a differentiable function of α; also

$$\sin\beta_1(\alpha) = r(\alpha)\sin\alpha/R(\beta_1(\alpha)),$$

$$\sin\beta_2(\alpha) = (1 - r(\alpha))\sin\alpha/R(\beta_2(\alpha)),$$

define β_1, β_2 as differentiable functions of α. Since $R \neq 0$, $\beta_1' \neq 0$, it follows that $R'(\beta_1(\alpha)) = 0$ if and only if $r'(\alpha)(r(\alpha) + a\cos\alpha) - ar(\alpha)\sin\alpha = 0$. Similarly $R'(\beta_2(\alpha)) = 0$ if and only if $r'(\alpha)(1 - r(\alpha) - a\cos\alpha) - a(1 - r(\alpha))\sin\alpha = 0$. Since $r'(\alpha) = r'(\pi - \alpha)$ it follows that $R'(\beta_1(\alpha)) = 0$ if and only if $R'(\beta_2(\alpha)) = 0$.

Now we observe that there is on C a point $Y(r, \alpha)$ determined uniquely from the equation $r(\alpha)\cos\alpha + a = 0$, for which $R'(\beta_1(\alpha)) = 0$ and $\pi/2 < \alpha < \pi$. Denoting the associated points by dashes, we have at $Y'(1 - r(\alpha), \pi - \alpha) \neq Y$: $R'(\beta_1) = R'(\beta_2) = 0$. From the symmetry properties of the curve we have $R'(\beta) = 0$ also at the point P_1 obtained by reflecting P across OY. We construct the sequence $\{P_i\}_1^\infty$ where $P_1 = Y'$, and for $i \geqq 2$, $P_i = Q_{i-1}'$, where Q_{i-1} is the point obtained by reflecting P_{i-1} across OY. The sequence $\{P_i\}_1^\infty$ converges to A, at every point P_i we have $R'(\beta_1(\alpha)) = R'(\beta_2(\alpha)) = 0$, and thus there are such points in the interval $[0, \beta_0]$. This leads to contradiction unless $a = 0$, in which case all the above conditions are trivially satisfied. This completes the proof.

The condition that C belongs to C^2 can be relaxed to $C \subset C^1$, as can be seen from the proof and from Dirac's treatment.

The author thanks the referee for clarifying suggestions.

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A NOTE ON THE SCHROEDER-BERNSTEIN THEOREM*

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Introduction. The proof of the Schroeder-Bernstein Theorem [1] is the critical step in the demonstration that the cardinal numbers are ordered. We may state the theorem as follows:

Let A and B be sets, and f, g be one-to-one mappings such that $f(A) \subset B$, $g(B) \subset A$; that is, f maps A onto a subset of B and g maps B onto a subset of A. Then there exists a one-to-one mapping F of A onto all of B.

To prove the theorem, we divide A into two parts, A_f and $A_{g^{-1}}$ and construct the mapping F of A onto B in such a way that $F=f$ on A_f and $F=g^{-1}$ on $A_{g^{-1}}$ [2]. The purpose of this note is to characterize all those sets A_f for which such a mapping F is one-to-one, and thus determine how inevitable is the choice of the set A_f used in the classical proof of the theorem.

1. Invariant subsets. We must begin with some preliminary observations about invariant sets. Let M be a set and let G be a one-to-one mapping of M onto itself. Then N is an invariant subset of M with respect to G whenever $G(N)=N$.

If $\{N_\alpha\}$ is an arbitrary collection of invariant subsets, then $\bigcup_\alpha N_\alpha$ and $\bigcap_\alpha N_\alpha$ are invariant subsets, since the mapping is one-to-one. Thus, there exists a minimal invariant subset (the intersection of all the invariant subsets), and a maximal invariant subset (the union of all the invariant subsets). Since the null set is invariant, it is the minimal invariant subset. The maximal invariant subset is $\bigcap_0^\infty G^k(M)$. To see this we observe that if N is any invariant subset, $G^k(N)=N$, so that $N=\bigcap_0^\infty G^k(N)$. Since $N \subset M$, $G^k(N) \subset G^k(M)$. Thus, $N = \bigcap_0^\infty G^k(N) \subset \bigcap_0^\infty G^k(M)$. Further, $\bigcap_0^\infty G^k(M)$ is itself invariant as is easily shown.

2. A Boolean equation. Now we shall show that the assertion that a certain Boolean equation has solutions implies that the Schroeder-Bernstein theorem is true.

If $F=f$ in A_f and $F=g^{-1}$ in $A_{g^{-1}}$, then for F to satisfy the conditions of the theorem, necessary and sufficient conditions are

(1) *g^{-1} must exist for all points of $A_{g^{-1}}$, so that $g(B) \supset A_{g^{-1}}$,*

(2) $$A = A_f + A_{g^{-1}},$$

(3) $$B = F(A_f) + F(A_{g^{-1}}) = fA_f + g^{-1}A_{g^{-1}}.$$

(By use of the $+$ instead of $\cup$ we imply that the sets involved are disjoint.)

Equation (3) shows that B is divided into two parts just as A is. That is, the mapping F^{-1} of B onto A is f^{-1} on fA_f and g on $g^{-1}A_{g^{-1}}$.

Let $g^{-1}A_{g^{-1}}=B_g$; thus (2), (3) become $A=A_f+gB_g$, $B=B_g+fA_f$, respectively. We can eliminate B_g and thus $A=A_f+g(B-f(A_f))=A_f+(gB-gfA_f)$, since g

* The authors wish to thank the referee for his valuable suggestions.

is a one-to-one mapping. Thus $A-A_f=gB-gfA_f$, since A_f and $gB-gfA_f$ are disjoint and $A_f \subset A$. Hence, $(A-A_f)+gfA_f=gB$, since gfA_f and $A-A_f$ are disjoint. Therefore,

$$gfA_f = gB - (A - A_f), \tag{4}$$

for the reason immediately above. Conversely, if A_f satisfies (4) and we set $B_g=B-fA_f$ then $A=A_f+gB_g$.

3. Solution of the equation. Inspection of (1), (2) and (4) shows that

$$A_f \supset A - g(B),\ A_f \supset S \textit{ implies that } A_f \supset gf(S).$$

From these two facts it follows by induction that $A_f \supset \underline{A}_f = \sum_0^\infty (gf)^k(A-g(B))$. (Here we have used $\sum$ instead of $\cup$ because the sets $(gf)^k(A-g(B))$ are all disjoint, a fact which is easily shown.)

It follows by direct substitution that $\underline{A}_f$ satisfies (1). If we let $A_f=\underline{A}_f+Q$, then, since gf is a one-to-one mapping, Q satisfies the equation $gf(Q)=Q$ so that Q is an invariant subset of A with respect to the mapping gf. Since we have used the $+$, the invariant subsets which we can use to construct solutions are, of necessity, disjoint from $\underline{A}_f$. However, this imposes no restriction on the class of invariant subsets because A_f is disjoint from $\bigcap_0^\infty (gf)^kA$, the maximal invariant subset, a fact which is easily shown.

We may summarize in the

THEOREM. *A_f satisfies equation* (1) *if and only if $A_f=\underline{A}_f+Q$ where $gf(Q)=Q$, and $\underline{A}_f=\sum_0^\infty (gf)^k(A-g(B))$.*

As an example, let A and B each be the positive integers and let f and g each be the successor function; *i.e.*, $f(N)=N+1$, $g(N)=N+1$. Then A_f is unique and consists of just the odd integers.

4. Inevitability of the choice of A_f. We see now that the set A_f to be used in the proof is unique up to the possible addition of a set invariant under the mapping gf. Since such an invariant set need not exist, the set A_f is the only set generally available for the proof.

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THE RANGE OF AN INTEGER-VALUED POLYNOMIAL

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It is a familiar and easily proven fact that a polynomial whose values at the integers are squares is the square of another polynomial. In this note we give a simple proof of the following generalization:

THEOREM. *Let $P(x)$ and $Q(x)$ be polynomials which are integer-valued at the integers, of degrees p and q respectively. If $P(n)$ is of the form $Q(m)$ for all n, or*

even for infinitely many blocks of consecutive integers of length $\geqq p/q+2$, *then there is a polynomial* $R(x)$ *such that* $P(x)=Q[R(x)]$.

Proof. Let $P(x)=a_px^p+\cdots+a_0, Q(x)=b_qx^q+\cdots+b_0$. Without loss of generality we may assume $a_p>0$ (for if $a_p<0$ we prove the theorem for the polynomials $-P$ and $-Q$). Also we may assume $P(n)=Q(m)$ for infinitely many blocks of *positive* integers n (in the contrary case, we would consider $P(-x)$ and $Q(-x)$). Finally, we can assume $b_q>0$. For, if q is even, this is necessarily the case, otherwise $Q(x)$ would take only finitely many positive values and so could not represent $P(n)$ for large positive n. If $Q(x)$ is odd, and $b_q<0$, we have only to replace $Q(x)$ by $Q(-x)$, which has the same range of values as $Q(x)$.

Now, let $x=\phi(w)$ denote a branch of the (algebraic) function inverse to the polynomial $w=Q(x)$, and indeed that branch for which $\phi(w)\sim(w/b_q)^{1/q}\to\infty$ as $w\to\infty$ along the positive real axis. $\phi(w)$ is positive and free of singularities if we restrict w to a suitable portion $w>w_0$ of the positive axis. Let now

$$f(x) = \phi[P(x)]. \tag{1}$$

Then $f(x)$ is a branch of an algebraic function, regular for $x>x_0$ and $\sim(a_p/b_q)x^{p/q}$ and we have $P(x)=Q[f(x)]$. Also, $f(n)$ is *integer-valued* at those values of n where $P(n)$ lies in the range of $Q(m)$, as we see from (1). Hence our theorem will be proved when we have proven the following

LEMMA. *Let* $f(x)$ *be a branch of an algebraic function real and regular for* $x_0<x<\infty$ *and satisfying* $|f(x)|<Cx^\alpha$ *where* $\alpha>0$, $C>0$. *If* $f(x)$ *is integer-valued for infinitely many blocks* $x=n, n+1, \cdots, n+r+1$, *where* r *is the least integer* $\geqq\alpha$, *then* f *is a polynomial.*

Proof of lemma. Suppose f is not a polynomial. Then $f^{(r+2)}(x)$ is a branch of an algebraic function, real for $x>x_0$, and $\not\equiv 0$. Hence $f^{(r+2)}(x)$ has only a finite number of zeros, and so $f^{(r+1)}(x)$ is monotonic (increasing or decreasing) for $x>x_1$ say. Therefore

$$\lim_{x\to\infty} f^{(r+1)}(x) = 0. \tag{2}$$

For, in the contrary case, we have either $f^{(r+1)}(x)\geqq a>0$ or $f^{(r+1)}(x)\leqq -a<0$ for $x\geqq x_2$. Integrating the first of these inequalities $r+1$ times from x_2 to x gives $f(x)>C_1x^{r+1}$ for large x, contradicting $f(x)<Cx^\alpha$. Similarly, the second alternative leads to a contradiction, and (2) is established.

Now, introducing the difference operator Δ, defined by $\Delta f(x)=f(x+1)-f(x)$, we know by the hypothesis that $\Delta^{r+1}f(n)$ is an integer for some arbitrarily large values of n. But by repeated application of the law of the mean, $\Delta^{r+1}f(x)=f^{(r+1)}(\xi)$ for some $x<\xi<x+r+1$. Hence for a sequence of $\xi_n\to\infty$, $f^{(r+1)}(\xi_n)$ is an integer, which must be zero for all large n because of (2). Thus $f^{(r+1)}(x)$, being an algebraic function vanishing at infinitely many points, is identically zero. Hence $f(x)$ is a polynomial, contrary to assumption. This completes the proof.

CLASSROOM NOTES

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INVARIANCE OF THE DISCRIMINANT OF A QUADRATIC FORM

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The following elementary demonstration of the invariance of the discriminant of a quadratic form avoids laborious computations and may have some merit for presentation to elementary calculus classes. Suppose that the quadratic form is given as $Ax^2+Bxy+Cy^2+Dx+Ey+F$ with discriminant $\Delta=B^2-4AC$. Let the angle through which the axes are rotated be α, and let the form now have coefficients $A(\alpha)$, $B(\alpha)$, $\cdots$, $F(\alpha)$ and discriminant $\Delta(\alpha) = B^2(\alpha) -4A(\alpha)C(\alpha)$. A convenient way of writing the new coefficients is

$$
\begin{aligned}
2A(\alpha) &= (A - C)\cos 2\alpha + B\sin 2\alpha + A + C,\\
B(\alpha) &= B\cos 2\alpha + (C - A)\sin 2\alpha,\\
2C(\alpha) &= (C - A)\cos 2\alpha - B\sin 2\alpha + A + C.
\end{aligned}
\tag{1}
$$

To show that $\Delta(\alpha)$ is independent of α, we will show that $\Delta'(\alpha)\equiv 0$, primes denoting differentiation with respect to α. From (1) it is easily verified that

$$
A'(\alpha) = B(\alpha), \qquad B'(\alpha) = 2[C(\alpha) - A(\alpha)], \qquad C'(\alpha) = -B(\alpha). \tag{2}
$$

Now $\Delta'(\alpha)=2B(\alpha)B'(\alpha)-4A(\alpha)C'(\alpha)-4A'(\alpha)C(\alpha)$ becomes, by (2),

$$
\Delta'(\alpha)\,B(\alpha)\{2\cdot 2[C(\alpha)-A(\alpha)]+4A(\alpha)-4C(\alpha)\}.
$$

Since this vanishes identically, the demonstration is complete.

BISECTOR OF AN INTERNAL ANGLE OF A TRIANGLE

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If the equations of the sides BC, CA, AB of a triangle ABC are $a_ix+b_iy+c_i=0$, $i=1, 2, 3$, then the internal bisector of, say, angle A is, of course,

$$
\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \pm \frac{a_3x + b_3y + c_3}{\sqrt{a_3^2 + b_3^2}} = 0, \tag{1}
$$

provided that the proper sign is chosen. This note establishes a simple criterion for making this choice.

The discussion will be simplified if the origin is shifted to the vertex A. Then the equations of the sides become $a_1x+b_1y+k=0$, $a_2x+b_2y=0$, $a_3x+b_3y=0$, respectively. The coordinates of B and C are

$$
\left(\frac{-kb_3}{a_1b_3 - a_3b_1}, \frac{ka_3}{a_1b_3 - a_3b_1}\right), \qquad \left(\frac{-kb_2}{a_1b_2 - a_2b_1}, \frac{ka_2}{a_1b_2 - a_2b_1}\right).
$$

The ratio of the lengths of the sides AB and AC is

$$\left| \sqrt{a_3^2+b_3^2}(a_1b_2-a_2b_1)/\sqrt{a_2^2+b_2^2}(a_1b_3-a_3b_1) \right|.$$

It follows that the coordinates of the point P which divides BC internally in this ratio are

$$\left[-k(b_2\sqrt{a_3^2+b_3^2} \pm b_3\sqrt{a_2^2+b_2^2})/D,\ k(a_2\sqrt{a_3^2+b_3^2} \pm a_3\sqrt{a_2^2+b_2^2})/D\right],$$

where

$$D=\sqrt{a_3^2+b_3^2}(a_1b_2-a_2b_1) \pm \sqrt{a_2^2+b_2^2}(a_1b_3-a_3b_1),$$

according as

$$(a_1b_2-a_2b_1)/(a_1b_3-a_3b_1) \gtrless 0. \tag{2}$$

Since AP is the angle bisector, inequality (2) is the criterion sought; it is invariant under a translation and so holds for the triangle referred to the original axes. Thus (1) and (2) jointly give the equation of the internal bisector of angle A, a result that offers a simple method for finding the incenter of a triangle when the equations of its sides are given.

AN APPLICATION OF THE MEAN VALUE THEOREM

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Given $y=u^v$, where u and v are differentiable functions of x, the derivative D_xy is found by logarithmic differentiation. Thus,

$$D_x(u^v)=vu^{v-1}\cdot D_xu+u^v\log u\cdot D_xv \tag{1}$$

is derived in this manner. In most textbooks, no attempt is made to derive (1) using the definition of the derivative, *i.e.*, $\lim_{\Delta x\to 0}\Delta y/\Delta x$. For in doing so, one runs into difficulties involving a binomial expansion.

The mean value theorem is usually introduced in the study of indeterminate forms. At this point, the derivative of u^v may be easily found. Thus

$$\Delta y=\left\{(u+\Delta u)^{v+\Delta v}-u^{v+\Delta v}\right\}+\left\{u^{v+\Delta v}-u^v\right\}. \tag{2}$$

If we apply the mean value theorem to the first term in (2), treating $v+\Delta v$ as fixed, we find that

$$(u+\Delta u)^{v+\Delta v}-u^{v+\Delta v}=(\Delta u)(v+\Delta v)\cdot r^{v+\Delta v-1},$$

where $u+\Delta u<r<u$ or $u<r<u+\Delta u$. Thus

$$\frac{\Delta y}{\Delta x}=(v+\Delta v)\cdot r^{v+\Delta v-1}\cdot\left(\frac{\Delta u}{\Delta x}\right)+\left(\frac{u^{v+\Delta v}-u^v}{\Delta v}\right)\cdot\left(\frac{\Delta v}{\Delta x}\right) \tag{3}$$

As $\Delta x\to 0$, $\Delta u\to 0$, $\Delta v\to 0$, $r\to u$, and (1) is now established.

A VECTOR PROOF OF EULER'S THEOREM ON ROTATIONS OF E^3

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A well known theorem of Euler states that every orientation preserving isometry T of Euclidean 3-space that has a fixed point O is a rotation about a line through O. The most difficult part of the proof of Euler's theorem is in showing that there exists a nonzero vector $\mathbf{P}$ such that $T(\mathbf{P})=\mathbf{P}$. We give a proof of this last statement which does not use the theory of matrices and determinants, and which may be presented to a class that is studying elementary vector algebra.

Let O be the origin, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be a right hand orthogonal system of unit vectors. We define $\mathbf{i}'=T(\mathbf{i})$, $\mathbf{j}'=T(\mathbf{j})$, $\mathbf{k}'=T(\mathbf{k})$. Since T is an isometry, $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ is an orthogonal system of unit vectors. Since T is orientation preserving, $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ is also a right hand system.

The following computation shows that the triple scalar product $[\mathbf{i}-\mathbf{i}', \mathbf{j}-\mathbf{j}', \mathbf{k}-\mathbf{k}']$ vanishes.

$$
\begin{aligned}
&[\mathbf{i}-\mathbf{i}', \mathbf{j}-\mathbf{j}', \mathbf{k}-\mathbf{k}'] \\
&\quad = [\mathbf{i}, \mathbf{j}, \mathbf{k}] - [\mathbf{i}, \mathbf{j}, \mathbf{k}'] - [\mathbf{i}, \mathbf{j}', \mathbf{k}] \\
&\qquad + [\mathbf{i}, \mathbf{j}', \mathbf{k}'] - [\mathbf{i}', \mathbf{j}, \mathbf{k}] + [\mathbf{i}', \mathbf{j}, \mathbf{k}'] + [\mathbf{i}', \mathbf{j}', \mathbf{k}] - [\mathbf{i}', \mathbf{j}', \mathbf{k}'] \\
&\quad = 1 - \mathbf{i} \times \mathbf{j} \cdot \mathbf{k}' - \mathbf{j}' \cdot \mathbf{k} \times \mathbf{i} + \mathbf{i} \cdot \mathbf{j}' \times \mathbf{k}' - \mathbf{i}' \cdot \mathbf{j} \times \mathbf{k} + \mathbf{j} \cdot \mathbf{k}' \times \mathbf{i}' + \mathbf{i}' \times \mathbf{j}' \cdot \mathbf{k} - 1 \\
&\quad = 1 - \mathbf{k} \cdot \mathbf{k}' - \mathbf{j}' \cdot \mathbf{j} + \mathbf{i} \cdot \mathbf{i}' - \mathbf{i}' \cdot \mathbf{i} + \mathbf{j} \cdot \mathbf{j}' + \mathbf{k}' \cdot \mathbf{k} - 1 = 0.
\end{aligned}
$$

It follows that the vectors $\mathbf{i}-\mathbf{i}'$, $\mathbf{j}-\mathbf{j}'$, $\mathbf{k}-\mathbf{k}'$ are coplanar, and that there exists a non-zero vector $\mathbf{P}$ which is orthogonal to each of them. Thus, $\mathbf{P}\cdot\mathbf{i}=\mathbf{P}\cdot\mathbf{i}'$, $\mathbf{P}\cdot\mathbf{j}=\mathbf{P}\cdot\mathbf{j}'$, $\mathbf{P}\cdot\mathbf{k}=\mathbf{P}\cdot\mathbf{k}'$.

Since T is an isometry and $T(O)=O$, T is clearly linear. That is, if x, y and z are real numbers, then $T(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=x\mathbf{i}'+y\mathbf{j}'+z\mathbf{k}'$. Thus,

$$
\begin{aligned}
T(\mathbf{P}) &= T((\mathbf{P}\cdot\mathbf{i})\mathbf{i} + (\mathbf{P}\cdot\mathbf{j})\mathbf{j} + (\mathbf{P}\cdot\mathbf{k})\mathbf{k}) = (\mathbf{P}\cdot\mathbf{i})\mathbf{i}' + (\mathbf{P}\cdot\mathbf{j})\mathbf{j}' + (\mathbf{P}\cdot\mathbf{k})\mathbf{k}' \\
&= (\mathbf{P}\cdot\mathbf{i}')\mathbf{i}' + (\mathbf{P}\cdot\mathbf{j}')\mathbf{j}' + (\mathbf{P}\cdot\mathbf{k}')\mathbf{k}' = \mathbf{P}.
\end{aligned}
$$

ON A GRAPHICAL SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS*

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In a previous note [1] the author has given a method for a graphical solution of the first order linear differential equation. An alternative construction is given by Betz, Burcham, and Ewing [2]. In this paper higher order linear differential equations will be solved graphically by first transforming them into an equivalent set of simultaneous first order equations.

A linear differential equation with constant coefficients can be factored into the form

* Presented to the American Mathematical Society, August 31, 1955.

$$[D - r_1][D - r_2] \cdots [D - r_n]y = F(x). \tag{1}$$

If we now let $[D-r_2] \cdots [D-r_n]y=S$, then $[D-r_1]S=F(x)$ and S then can be solved for graphically [1]. Consequently, (1) has been reduced in degree by one. Continuing in this fashion, we can determine y. In case some of the roots $\{r_n\}$ are complex, we can determine an alternative real factorization.

For example, let us consider

$$[D^2 - 2aD + a^2 + b^2]y \equiv [D - a - bi][D - a + bi]y = F(x). \tag{2}$$

We can obtain a real factorization by letting $[D^2-2aD+a^2+b^2] \equiv [D-m] \times [D-n]$, where m and n are functions of x. Thus,

$$m + n = 2a, \qquad mn - n' = a^2 + b^2. \tag{3}$$

Solving (3) simultaneously, we find that

$$[D^2 - 2aD + a^2 + b^2]y \equiv [D - a - b \tan bx][D - a + b \tan bx]y. \tag{4}$$

Incidentally, by means of the exponential shift theorem, this factorization leads immediately to the closed form solution,

$$y = e^{ax} \cos bx \int \frac{dx}{\cos^2 bx} \int e^{-ax} F(x) \cos bx dx, \tag{5}$$

of (2). By integrating (5) by parts, we can obtain the solution in the form one would obtain by the method of variation of parameters.

If, in (1), each r_n is a function of x, the solution is carried through in the same manner as indicated.

Let us now consider the general second order linear differential equation, $[D^2+2A(x)D+B(x)]z=G(x)$. As this equation cannot, in general, be factored, we have to resort to a different type of reduction. First, the equation is reduced to the canonical form,

$$[D^2 + \phi(x)]y = F(x), \tag{6}$$

by means of the transformation $y=z \exp(\int A dx)$. We now factor the operator $[D^2+\phi(x)]$ into the form $[D+m][D+n]$. Consequently, m and n must satisfy the two equations

$$m + n = 0, \qquad mn + n' = \phi(x). \tag{7}$$

Eliminating m, n must satisfy the Riccati equation

$$dn/dx = n^2 + \phi(x). \tag{8}$$

Thus (8), taken simultaneously with

$$[D - n][D + n]y = F(x), \tag{9}$$

is equivalent to (1). The transformation of (6) into (8) and (9), differs from the

usual transformation (*i.e.* by letting $dy/dx=p$) in that the dependent variables have been separated. This method of reduction may even be advantageous in some cases for a numerical solution.

To get a graphical solution of (8), we let $2n=p^2$. Then, $dp/dx=[p^2+4\phi(x)]/[4p]$, and we use the method [1] of the author for equations of the form $dy/dx=[\phi(x)-G(y)]/[\psi(x)-F(y)]$. Another method would be to plot the two curves C_1: $[x=n^2]$, and C_2: $[n=\phi(x)]$, as shown in Figure 1. The lines P_0A and P_0D

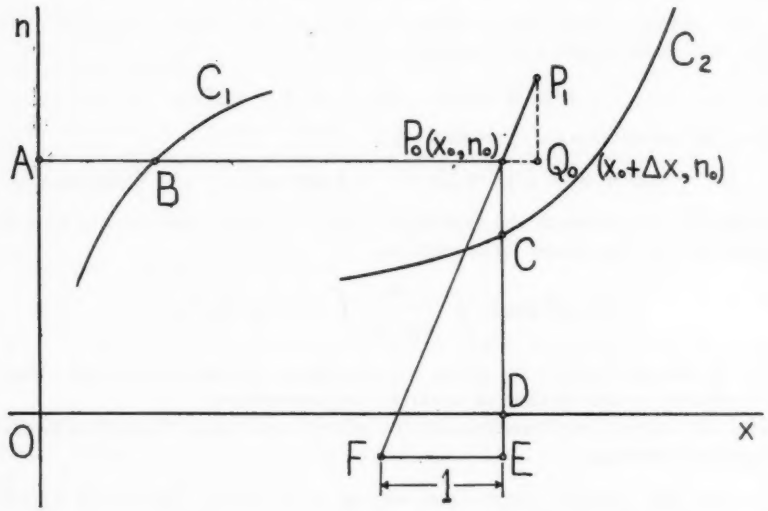

FIG. 1

are then drawn parallel to the axes. It then follows that $AB=n_0^2$ and $CD=\phi(x_0)$. Using a pair of dividers, we lay off P_0E which is the sum of AB and CD. The slope of line FP_0 is then $\phi(x_0)+n_0^2$. Thus, if P_0Q_0 represents Δx, P_1 is an approximation to a point on the integral curve of (8) through the point (x_0, n_0), n_0 being arbitrary. The procedure is then repeated from point P_1. The proximity of the successive points $P_0, P_1, \cdots$, *etc.*, will determine the accuracy of the construction. Equation (9) is now solved in a manner similar to that for (1); by reducing it to the two linear equations,

$$[D-n]S=F(x), \tag{10}$$

$$[D+n]y=S. \tag{11}$$

If the boundary conditions for (1) are $x=x_0$, $y=y_0$, $y'=y_1$, then the boundary condition for (10) is $x=x_0$, $S=y_1+n(x_0)y_0$.

If we have other boundary conditions, for example, the two point conditions

$x=x_0$, $y=y_0$; and $x=x_1$, $y=y_1$, we first determine two solutions $y=F_1(x)$, and $y=F_2(x)$ satisfying $F_1(x_0)=y_0$, $F_1'(x_0)=m_1$; $F_2(x_0)=y_0$, and $F_2'(x_0)=m_2$ (m_1 and m_2 are arbitrary). Then $y=aF_1(x)+bF_2(x)$, where $a+b=1$, $aF_1(x_1)+bF_2(x_1)=y_1$.

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NOTE ON INTEGRATION BY OPERATORS

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Certain convenient devices for performing indefinite integrations may be found in the consideration of differential operators. The validity of the results, and of intermediate steps, may be established by differentiation procedures. Such devices as these can be presented to students of integral calculus, and they are especially useful to students of differential equations. The three indefinite integrations (particular integrals) which are shown here are illustrative of several which can be performed easily by operator methods.

The first is

$$\int e^{ax}\sin bx\,dx = \frac{1}{D}e^{ax}\sin bx = e^{ax}\frac{1}{D+a}\sin bx$$

$$= e^{ax}\frac{D-a}{D^2-a^2}\sin bx = \frac{e^{ax}}{-b^2-a^2}(D-a)\sin bx$$

$$= \frac{e^{ax}}{-b^2-a^2}(b\cos bx - a\sin bx).$$

Similarly,

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{-b^2-a^2}(-b\sin bx - a\cos bx).$$

A third is

$$\int x^n e^{ax}dx = \frac{1}{D}e^{ax}x^n = e^{ax}\frac{1}{D+a}x^n = e^{ax}\frac{1}{a}\left(1+\frac{D}{a}\right)^{-1}x^n$$

$$= \frac{e^{ax}}{a}\left(1-\frac{D}{a}+\frac{D^2}{a^2}-\frac{D^3}{a^3}+\cdots\right)x^n$$

$$= \frac{e^{ax}}{a}\left[x^n-\frac{nx^{n-1}}{a}+\frac{n(n-1)x^{n-2}}{a^2}-\cdots+(-1)^{n-1}\frac{n!}{a^n}\right].$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1271. *Proposed by Ward Cheney and A. A. Goldstein, Convair-Astronautics, San Diego, California*

In E_3 let L denote a line not through the origin and not parallel to any coordinate plane. Does the point of contact of L with the smallest sphere having center at the origin necessarily lie in the same octant as the point of contact of L with the smallest cube having center at the origin and edges parallel to the coordinate axes?

E 1272. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If A, B, C are the angles of a triangle, show that

$$(\sin A/2+\sin B/2+\sin C/2)^2 \leqq \cos^2 A/2+\cos^2 B/2+\cos^2 C/2.$$

E 1273. *Proposed by Alan Wayne, Baldwin, New York*

Prove or disprove the following proposition (which is true for $n=3$ and $n=4$): A plane n-gon with incircle of radius r and circumcircle of radius R is regular if and only if $r=R\cos(\pi/n)$.

E 1274. *Proposed by P. G. Kirmser, Kansas State College*

Given $p_i>0$, $q_i>0$, and $\sum_{i=1}^{n} p_i=\sum_{i=1}^{n} q_i$, show that

$$\sum_{i=1}^{n} p_i \ln p_i \geqq \sum_{i=1}^{n} p_i \ln q_i.$$

E 1275. *Proposed by M. S. Klamkin, AVCO Research and Advanced Development, Lawrence, Massachusetts*

Solve for x:

$$\int_0^x s^{3/3}(1-s)^{4/3}ds = \int_0^1 t^{3/3}/(1+t)^{-6}dt.$$

SOLUTIONS

A Holiday Greeting

E 1241 [1956, 723]. *Proposed by C. W. Trigg, Los Angeles City College*

The holiday greeting, *MERRY XMAS TO ALL*, is a cryptarithm in which

each of the letters is the unique representation of a digit, and each word is a square integer. Find all solutions.

Solution by L. A. Ringenberg, Eastern Illinois State College. We solve the problem using a table of squares as follows. Note that $ALL=100$, 144, 400, or 900. If $ALL=100$ or 144, then $XMAS=2916$ or 9216 and TO is not a square. If $ALL=400$, then $XMAS=1849$, 3249, 6241, or 8649. If $ALL=400$ and $XMAS=1849$, then $M=8$, $MERRY=81225$ and $E=X$. If $ALL=400$ and $XMAS=3249$, then $M=2$, $MERRY=27556$, and $TO=81$; hence the solution

(I) 27556 3249 81 400.

If $ALL=400$ and $XMAS=6241$, then TO is not a square. If $ALL=400$ and $XMAS=8649$, then $M=6$, $L=0$, and there is no solution for $MERRY$. If $ALL=900$, then $XMAS=1296$ or 7396. If $ALL=900$, $XMAS=1296$, then TO is not a square. If $ALL=900$, $XMAS=7396$, then $M=3$, $MERRY=34225$, and $TO=81$; hence the solution

(II) 34225 7396 81 900.

(I) and (II) are all of the solutions of the cryptarithm.

Also solved by A. W. Adler and F. C. Koch (jointly), Leon Bankoff, H. F. Bennett, Julian Braun, D. A. Breault, Dori Burke, Dorothy I. Carpenter, J. F. Daly, J. E. Darraugh, Monte Dernham, Underwood Dudley, Hazel E. Evans, H. M. Gehman, Basil Gordon and H. E. Sturgis (jointly), A. G. Grace, Jr., A. R. Hyde, Edgar Karst, C. D. Keim, H. R. Kingston, J. M. Kingston, J. D. E. Konhauser, Sidney Kravitz, A. M. Linn, Joe Lipman, R. L. London, D. C. B. Marsh, Erich Michalup, Herbert Nadler, C. S. Ogilvy, Marghrita L. Oneil, Bart Park, D. S. Passman, P. W. A. Raine, Azriel Rosenfeld, Frank Saunders, E. D. Schell, F. M. Stein, Alan Wayne, K. G. Whyburn, R. E. Wyllys, David Zeitlin, and the proposer. Late solutions by R. K. Guy and Tadao Shingu.

A number of the above solutions were incomplete.

Rosenfeld observed that if the further requirement is imposed that the *sum of the digits* of each word be a perfect square, then the solution is unique. Karst concocted the allied problem: The holiday greeting, $MERRY+XMAS=TOALL$, is a cryptarithm in which each of the letters is the unique representation of a digit and each word is divisible by 3. There is now a unique solution, $84771+5862=90633$.

A Circle Concentric with a Nine Point Circle

E 1242 [1956, 724]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the circle orthogonal to the circles inscribed in the squares of centers A', B', C' constructed exteriorly (or interiorly) on the sides of a triangle ABC is concentric with the nine point circle of triangle $A'B'C'$.

Solution by the proposer. Denote the lengths of the sides BC, CA, AB of triangle ABC by a, b, c. Let O'' be the midpoint of $B'C'$; let N'' be the intersection with $B'C'$ of the radical axis $\Delta_{B'C'}$ of the two circles $(B', b/2)$, $(C', c/2)$; let A'' be the foot of the altitude $A'A''$ of triangle $A'B'C'$. Then we easily find

(1) $$(N''B')^2 - (N''C')^2 = (b^2 - c^2)/4 = 2(B'C')(O''N'')$$

and, by using the law of cosines on triangles $A'CB'$ and $A'BC'$,

$$(2) \qquad (A'B')^2 = (a^2 + b^2)/2 + 2S, \qquad (A'C')^2 = (a^2 + c^2)/2 + 2S,$$

where S represents the area of triangle ABC. From (2) we find

$$(A'B')^2 - (A'C')^2 = (b^2 - c^2)/2 = 2(B'C')(O''A''),$$

whence, by comparison with (1), $O''N'' = (O''A'')/2$. It follows that $\Delta_{B'C'}$ passes through the nine point center N' of triangle $A'B'C'$. Similarly, $\Delta_{C'A'}$ and $\Delta_{A'B'}$ pass through N'. But the point of concurrence of the three radical axes of three circles taken in pairs is the center of the circle cutting the three circles orthogonally.

The proof can be modified to take care of the case of interiorly constructed squares.

Also solved by J. W. Clawson (essentially as above) and D. C. B. Marsh (analytically). Late solutions by R. K. Guy and Josef Langr.

Product of a Number and Its Reverse

E 1243 [1956, 724]. *Proposed by M. A. Rashid and M. A. Uppal, Lahore, West Pakistan*

Prove that the product of a number consisting of two digits and its reverse is never a square except when the two integers are equal.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* Let the two-digit number be written as $10t+u$ (with t, u non-negative digits less than 10); its product with its reverse is

$$10t^2 + 101tu + 10u^2 = 10(t - u)^2 + 121tu \equiv 10(t - u)^2 \pmod{11}.$$

Since 10 is a quadratic nonresidue, modulo 11, it follows that $t-u \equiv 0 \pmod{11}$ and $t=u$.

II. *Solution by Alan Wayne, Baldwin, N. Y.* Let the number be $10t+u$, where t is the tens digit and u is the units digit. Consider $(10t+u)(10u+t)=k^2$, to be solved in positive integers. Obviously, when $t=u$, then $k=11t$ defines solutions. Suppose $t>u$ and let $x=t+u$, $y=t-u$. Then $(9y)^2+(2k)^2=(11x)^2$, an equation whose primitive solutions are

$$9y = p^2 - q^2, \qquad 2k = 2pq, \qquad 11x = p^2 + q^2,$$

where p and q are integers of opposite parity and relatively prime. It follows that $10t+u=p^2$, and $10u+t=q^2$. However, there are no two-digit squares ($t \neq u$) whose reversals are also squares. This completes the proof.

The property does not extend to three or four-digit numbers, as is shown by the examples $(961)(169)=(403)^2$ and $(9801)(1089)=(3267)^2$. There arises the following conjecture for proof or disproof: When an integer and its reversal are unequal, their product is never a square except when both are squares.

Also solved by W. A. Al-Salam, Leon Bankoff, H. F. Bennett, W. J. Blundon, D. A. Breault, J. E. Darraugh, J. E. D'Atri, Underwood Dudley, Hazel E. Evans, A. R. Hyde, I. M. Isaacs, J. H. Kaplan, M. S. Klamkin, J. J. Kohn, Joe Lipman, J. W. Mettler, Herbert Nadler, C. S. Ogilvy, D. S. Passman, Angelo Perlis, L. A. Ringenberg, D. A. Robinson, Azriel Rosenfeld, Gustavus Simmons, Art Steger, H. S. Valk, and David Zeitlin. Late solution by R. K. Guy.

Dudley solved the problem by actually enumerating and testing the 45 possible products. Some of the submitted solutions were open to grave objections.

A Series for Pi

E 1244 [1956, 724]. *Proposed by Leo Moser, University of Alberta*

Show that

$$\sum_{n=0}^{\infty} [5/(7n+1)(7n+6) + 3/(7n+2)(7n+5) - 1/(7n+3)(7n+4)] = \pi/\sqrt{7}.$$

Solution by A. E. Danese, Mt. Morris, N. Y. The given series can be written as

$$S = \sum_{n=0}^{\infty} [1/(7n+1) + 1/(7n+2) - 1/(7n+3) + 1/(7n+4) - 1/(7n+5) - 1/(7n+6)].$$

The expansion in partial fractions of the cotangent,

$$\pi t \cot \pi t = 1 + 2t^2 \sum_{k=1}^{\infty} 1/(t^2 - k^2), \qquad t \neq k,$$

with $t = x/7$ becomes

$$\phi(x) = (\pi/7) \cot (\pi x/7) = 1/x + \sum_{k=1}^{\infty} [1/(x - 7k) + 1/(x + 7k)].$$

Then $S = \phi(1) + \phi(2) - \phi(3) = \pi/\sqrt{7}$.

The series S occurs in Knopp, *Infinite Series*, p. 268, Ex. 108 (d).

Also solved by Leonard Carlitz, N. J. Fine, Emil Grosswald, F. W. Herlihy, A. R. Hyde, M. S. Klamkin, Erich Michalup, Chih-yi Wang, and David Zeitlin. Late solution by R. K. Guy.

This problem was published in *Newsletter of the Canadian Mathematical Congress*, 2nd issue, Autumn, 1954, by the same proposer. For an allied problem proposed by Chih-yi Wang, see Problem 287 in the 1956 Nov.–Dec. issue of *Mathematics Magazine*.

Convergence of Two Associated Sequences

E 1245 [1956, 724]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If

$$b_{n+1} = \int_0^1 \min(x, a_n)\,dx, \qquad a_{n+1} = \int_0^1 \max(x, b_n)\,dx,$$

prove that the sequences $\{a_n\}$ and $\{b_n\}$ both converge and find their limits.

Solution by N. J. Fine, University of Pennsylvania. For any a_0, b_0, it is easy to see that a_n and b_n both lie between 0 and 1 for all $n \geqq 2$. The recurrence formulas then become (for $n \geqq 2$)

$$a_{n+1} = (1 + b_n^2)/2, \qquad b_{n+1} = a_n - a_n^2/2.$$

If we assume that $\lim a_n = a$, $\lim b_n = b$, they must satisfy

$$a = (1 + b^2)/2, \qquad b = a - a^2/2,$$

from which we get

$$a + b - 1 = (a + b - 1)(b - a + 1)/2.$$

Since the factor $(b-a+1)/2 \neq 1$, we have $a+b=1$, and this yields $a = 2-\sqrt{2}$, $b = \sqrt{2}-1$. To show that $a_n \to a$, $b_n \to b$, we write $a_n = a+\delta_n$, $b_n = b+\epsilon_n$. The recurrence formulas become, after an easy reduction,

$$\delta_{n+1} = (b + \epsilon_n/2)\epsilon_n, \qquad \epsilon_{n+1} = (b - \delta_n/2)\delta_n.$$

Now $|\epsilon_n| = |b_n - b| \leqq \max(b, 1-b) = a$, and $|\delta_n| = |a_n - a| \leqq \max(a, 1-a) = a$. Hence $|b+\epsilon_n/2| \leqq b + a/2 = 1/\sqrt{2}$, $|b-\delta_n/2| \leqq b + a/2 = 1/\sqrt{2}$. Therefore $|\delta_{n+1}| \leqq |\epsilon_n|/\sqrt{2}$ and $|\epsilon_{n+1}| \leqq |\delta_n|/\sqrt{2}$. This shows that $\delta_n \to 0$, $\epsilon_n \to 0$, and the proof is complete.

Also solved by Nathaniel Grossman, Emil Grosswald, A. R. Hyde, P. G. Kirmser, D. C. B. Marsh, Mary Payne, Lawrence Shepp, L. K. Williams, and the proposer. Late solution by R. K. Guy.

A number of the submitted solutions failed to show convergence.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4743. *Proposed by Aaron Herschfeld, Canisius College, Buffalo, N. Y.*

Given an arbitrary positive number u_0, consider the sequence $\{u_n\}$, where $u_n = u_{n-1} + 1/u_{n-1}$. Prove the relation $u_n = \sqrt{2n} + (\log n)/\sqrt{32n} + O(1/\sqrt{n})$.

4744. *Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.*

Three congruent ellipses are mutually tangent. Determine the maximum and the minimum of the area bounded by the three ellipses.

4745. *Proposed by Marvin Marcus, National Bureau of Standards and the University of British Columbia*

Let $A(t)$ be a complex n-square matrix function, continuous on $[0, \infty)$. Let A^* be the conjugate transpose of A and let $X(t)$ be the real part of the trace of $A(t)$. Assume that

$$\text{(i)} \qquad A(t) + A^*(t) \geqq 0, \quad 0 \leqq t < \infty; \qquad \text{(ii)} \quad \limsup_{t \to \infty} \int_0^t X(s)ds < \infty.$$

Show that all solutions of the linear vector-matrix differential equation $\dot{y} = A(t)y$ are bounded on $[0, \infty)$.

4746. *Proposed by Leonard Carlitz, Duke University*

Let p be an odd prime. Show that the number of solutions of the system

$$\text{(1)} \qquad x + y + z \equiv 3, \qquad xyz \equiv 1 \pmod{p}$$

is equal to $p - 2 - (-3|p)$, where $(-3|p)$ denotes the quadratic character of -3. If $p > 3$, show also that the number of solutions is the same for the system

$$\text{(2)} \qquad x + y + z \equiv 3, \qquad x^3 + y^3 + z^3 \equiv 3 \pmod{p}.$$

4747. *Proposed by B. J. Boyer, Lafayette, Ind.*

Write $S_n = \sum_{k=0}^{n} B_k^2$, where the B_k are Fibonacci numbers, $B_0 = 1$, $B_1 = 1$, $B_{n+2} = B_{n+1} + B_n$. Find the value of $\sum_{n=0}^{\infty} (-1)^n / S_n$.

SOLUTIONS

Simson Line and Euler Line

4695 [1956, 426]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a cyclic quadrangle the Simson line of one vertex with respect to the triangle formed by the other three is perpendicular to the Euler line of that triangle, then the same property holds for the other vertices of the quadrangle.

Solution by Sister M. Stephanie, Georgian Court College, Lakewood, N. J. Using complex coordinates and taking the circle to be the unit circle, let the vertices of the quadrangle be t_1, t_2, t_3, t_4, $|t_i| = 1$. The Simson line of t_1, with respect to the triangle formed by t_2, t_3, t_4, has the equation

$$2t_1^2\bar{z} - 2t_1s_3\bar{z} + s_3 + t_1s_2 - t_1^3 - t_1^2s_1 = 0,$$

where s_1, s_2 and s_3 are the elementary symmetric functions of t_2, t_3 and t_4. The Euler line of triangle t_2, t_3, t_4 has the equation $s_2z - s_1s_3\bar{z} = 0$. If the two lines are perpendicular, one clinant is the negative of the other, whence $s_3/t_1 = -s_1s_3/s_2$, or upon simplifying, $t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=0$. The symmetry of this result guarantees that the property holds equally for any vertex.

Also solved by J. W. Clawson, G. W. Courter, R. Deaux, Beckham Martin, O. J. Ramler, Robert Sibson, Chih-yi Wang, and the proposer.

Mangoldt's Function

4696 [1956, 426]. *Proposed by T. M. Apostol, California Institute of Technology*

Let $\Lambda(n)$ denote Mangoldt's function, defined as follows:

$$\Lambda(n) = \begin{cases} \log p, \text{ if } n \text{ is a power of the prime } p, \\ 0, \text{ otherwise.} \end{cases}$$

For $n>1$, let $f_m(n)$ denote the number of representations of n as a product of m factors, where each factor is greater than 1 and the order of the factors is taken into consideration (*e.g.*, $f_2(24)=6$ and $f_3(16)=3$). Prove the identity

$$\Lambda(n) = (\log n) \sum_{m=1}^{\nu(n)} \frac{(-1)^{m+1}}{m} f_m(n) \qquad (n > 1),$$

where $\nu(n)$ is the total number of prime factors of n, *i.e.*, $\nu(n)=a_1+a_2+\cdots$, if $n=p_1^{a_1}p_2^{a_2}\cdots$.

Solution by J. V. Whittaker, University of California, Los Angeles. The generating functions for $\Lambda(n)$ and $f_m(n)$ are

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \qquad (\zeta(s) - 1)^m = \sum_{n=1}^{\infty} \frac{f_m(n)}{n^s}.$$

The first of these is well known, and the second can easily be obtained if we agree that $f_m(n)=0$ when $m>\nu(n)$. Differentiating the second equation and rearranging factors, we have

$$\sum_{n=1}^{\infty} \frac{(\log n) f_m(n)}{mn^s} = -(\zeta(s) - 1)^{m-1}\zeta'(s).$$

Multiplying both sides by $(-1)^{m+1}$ and summing from $m=1$ to ∞, we find that

$$\sum_{n=1}^{\infty} \frac{\log n}{n^s} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} f_m(n) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

The convergence of the series is assured, say, for $s \geqq 2$. The desired result now

follows from the uniqueness of the Dirichlet series representation.

Also solved by Leonard Carlitz, Nathaniel Grossman, J. B. Johnston, C. D. Olds, Abe Sklar, and the proposer.

Generalization of De Moivre's Quintic

4697 [1956, 427]. *Proposed by Leonard Carlitz, Duke University*

Find the roots of the equation

$$y^n - ny^{n-2} + \frac{n(n-3)}{2!}y^{n-4} + \cdots + (-1)^r \frac{n(n-r-1)\cdots(n-2r+1)}{r!}y^{n-2r} + \cdots = 2a,$$

in particular, when $a=1$.

Solution by D. C. B. Marsh, Colorado School of Mines. Let $y=2\cos\theta$, and the given equation reduces to $\cos n\theta = a$ by a familiar result (See, *e.g.*, Chrystal, *Textbook of Algebra*, II, p. 276, (7), where the result is credited to James Bernoulli.) The roots are therefore $y=2\cos\{(2\pi k+\text{Arc cos } a)/n\}$, $k=1, \cdots, n$.

Also solved by H. W. Gould, C. D. Olds, Chih-yi Wang, and the proposer.

Editorial Note. In Dickson, *First Course in the Theory of Equations*, p. 139, a solution is indicated for the equation

$$y^n - nqy^{n-2} + \frac{n(n-3)}{1\cdot 2}q^2y^{n-4} - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3}q^3y^{n-6} + \cdots = c,$$

of which the present problem is a particular case. The values of y are given by $A+B$ and $A\epsilon^j+B\epsilon^{n-j}$, $j=1, \cdots, n-1$, where A is one value of $\{(c+\sqrt{c^2-4q^{2k}})/2\}^{1/k}$, $B=q/A$, and ϵ is a primitive nth root of unity. See also problems E 223 [1937, 109] and 4568 [1955, 189].

An Extension of a Result of Hilbert

4698 [1956, 497]. *Proposed by H. F. Sandham, Dublin Institute for Advanced Studies, Ireland*

Prove that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m a_n}{m+n+1/6} \leqq 2\pi\sum_{r=0}^{\infty}a_r^2,$$

provided the right-hand side converges.

Editorial Note. As pointed out by several correspondents, this problem is a special case of III, 169, Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, pp. 117, 290, with slight change in notation.

Solved by Leonard Carlitz, N. J. Fine, Basil Gordon, Peter Henrici, J. Horváth, Marvin Rosenblum, Chih-yi Wang, and the proposer.

4-, 5-, 6-, and 7-Lines

4700 [1956, 498]. *Proposed by J. W. Clawson, Ursinus College*

It is well known that the midpoints of the diagonals of a 4-line lie on a straight line which has been called the *Newtonian* of the 4-line. It is also known that the Newtonians of the five 4-lines obtained by omitting in turn each of the sides of a 5-line concur in a point which we may call the *Newtonian point* of the 5-line.

Prove (1) that the Newtonian points of the six 5-lines obtained by omitting in turn each of the sides of a 6-line lie on a conic which may be called the *Newtonian conic* of the 6-line; (2) that the Newtonian conics of the seven 6-lines obtained by omitting in turn each of the sides of a 7-line concur in three points, two of which may be imaginary.

Solution by Roland Deaux, Faculté Polytechnique, Mons, Belgium. We suppose that no two of the sides 1, 2, 3, 4, 5, 6, 7 of a 7-line are parallel, that no three of them are concurrent, and that no six are tangent to a conic. By omitting for instance 6, 7, we obtain the 5-line 12345 and the five 4-lines 2345, 1345, 1245, 1235, 1234 whose Newtonians n_{67}^1, n_{67}^2, n_{67}^3, n_{67}^4, n_{67}^5 concur in the Newtonian point N_{67} of the 5-line.

Such notations as $n_{67}^1 \equiv n_{76}^1$, n_{17}^6, n_{16}^7 clearly show the identity of those three lines. The Newtonian points N_{67}, N_{17}, N_{16} are thus collinear.

We have proved [*Bulletin de l'Institute Polytechnique de Jassy*, vol. 7, 1948, p. 28] that the points at infinity of the pairs of lines

$$(1, \overset{1}{n_{67}}), \quad (2, \overset{2}{n_{67}}), \quad (3, \overset{3}{n_{67}}), \quad (4, \overset{4}{n_{67}}), \quad (5, \overset{5}{n_{67}})$$

are five pairs of a quadratic involution $\dot{I}$ (See also Remark 1, below.) A similar property holds for the pairs

$$(1, \overset{1}{n_{57}}), \quad (2, \overset{2}{n_{57}}), \quad (3, \overset{3}{n_{57}}), \quad (4, \overset{4}{n_{57}}), \quad (6, \overset{6}{n_{57}}).$$

The pencils $(n_{67}^1 n_{67}^2 n_{67}^3 n_{67}^4)$, $(n_{57}^1 n_{57}^2 n_{57}^3 n_{57}^4)$ being homographic, their centers N_{67}, N_{57} and the common points N_{17}, N_{27}, N_{37}, N_{47} of any two corresponding rays lie on the Newtonian conic ν_7 of the 6-line 123456. Hence the first part of the problem.

We now consider the Newtonian conic $\nu_6 \equiv N_{16}N_{26}N_{36}N_{46}N_{56}N_{76}$ of the 6-line 123457 which cuts ν_7 in N_{67} and in three points A, B, C, two of which may be imaginary. The lines n_{67}^1, n_{67}^2, n_{67}^3, n_{67}^4, n_{67}^5 pass through N_{67}, meet ν_6 in N_{16}, N_{26}, N_{36}, N_{46}, N_{56} and meet ν_7 in N_{17}, N_{27}, N_{37}, N_{47}, N_{57}.

Since the ranges

$$(N_{16}, N_{26}, N_{36}, N_{46}, N_{56}, A, B, C), \quad (N_{17}, N_{27}, N_{37}, N_{47}, N_{57}, A, B, C)$$

are projectively related, so are such pencils as

$$N_{16}(N_{26}, N_{36}, N_{46}, N_{56}, A, B, C), \qquad N_{17}(N_{27}, N_{37}, N_{47}, N_{57}, A, B, C).$$

Their centers N_{16}, N_{17} and the common points N_{12}, N_{13}, N_{14}, N_{15}, A, B, C of any two corresponding rays lie therefore on the Newtonian conic ν_1 of the 6-line 234567. Hence the second part of the problem.

Remarks. 1. The existence of involution I may also be established as follows. In the 5-line 12345 let M_1, M_2, M_3 be the midpoints of the diagonals joining the vertex 45 with the vertices 23, 31, 12. We have $N_{67}M_1 = n^1_{67}$, $N_{67}M_2 = n^2_{67}$, $N_{67}M_3 = n^3_{67}$ and the lines M_2M_3, M_3M_1, M_1M_2 are respectively parallel to the sides 1, 2, 3. The three pairs of opposite sides of the quadrangle $N_{67}M_1M_2M_3$ are cut by the line at infinity in three pairs of points belonging to I. The same reasoning applied to the vertices 35, 25 and the remaining triangles 124, 134 will prove the theorem.

2. In the 5-line 12345 let P be the symmetric of the vertex 45 with respect to the point N_{67}. The quadrangle whose vertices are P and those of triangle 123 may replace the quadrangle $N_{67}M_1M_2M_3$ for the determination of I. The unit points of I are thus cyclic or on two rectangular directions according as P is the orthocenter of triangle 123 or lies on the circumcircle. Hence the theorem:

Consider a 5-line circumscribed to an ellipse or a hyperbola with center 0. *If the symmetric with respect to* 0 *of a vertex is the orthocenter of the triangle formed by the sides which do not contain that vertex or lies on the circumcircle of that triangle, the same property holds for the other* 9 *vertices.*

Also solved by the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. III: *Astronomy and Physics*. Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956. ix+252 pp. $6.25.

The portion of the Third Berkeley Symposium on Mathematical Statistics and Probability reported in this volume of the *Proceedings* was held in December, 1954, in conjunction with the meeting in Berkeley of the AAAS. The papers here reported were given in sessions jointly sponsored by mathematicians, astronomers, and physicists. There are eight papers dealing with astronomy, and

five with physics. They are of uniformly high quality; but it is natural to expect that the papers of greatest interest to the readers of this MONTHLY will be the papers with considerable mathematical interest, including the five on mathematical physics.

Statistical Mechanics and Probability Theory by André Blanc-Lapierre and Albert Tortrat is the first of this quintet. It deals with those problems of statistical mechanics that stem from the fact that the systems considered have many degrees of freedom. The authors point out that the reduction of the problems of statistical mechanics to probability theory results more from the similarity of the mathematical expressions than from the nature of the problems. The results are interpreted for Bose-Einstein, for Fermi-Dirac, and for classical statistics. Both the method of probability density and the method of characteristic functions are used. *Foundations of Kinetic Theory* by M. Kac has a superb quality of exposition. It re-examines critically the derivation of the reduced Boltzmann equation and the conclusions drawn from it. There are clear indications of the limitations in the validity of results and of theorems needing proof. As the author says, more questions are raised than answered. In *Random Solutions of Partial Differential Equations* by J. Kampé de Fériet, the author considers boundary value problems for linear partial differential equations of elliptic and parabolic type, but replaces the function $f(A)$ defining the boundary value by a random function $f_\omega(A)$ where ω represents a point in a suitable probability space. This point of view is motivated by physical considerations. The limitation of the discussion to one elliptic equation in a finite domain, and one parabolic equation in an infinite domain is accepted because of the insurmountable difficulties in applying the "random" point of view to the Navier-Stokes equations corresponding to a given random velocity field at time $t=0$.

The *Theory of the Vibration of Simple Cubic Lattices with Nearest Neighbor Interactions* by Elliott W. Montroll is a powerful analysis of two of the problems discussed and a more brief handling of the third and fourth. There are two appendices dealing with extensive proofs of mathematical theorems needed in the body of the paper. Some of the mathematical problems have application in the theory of random walks on discrete lattices.

In *Nonlinear Prediction and Dynamics*, Norbert Wiener gives a most interesting and suggestive treatment of the theory of prediction in the nonlinear case, and related matters. There are quotable passages about the nature and role of hypothesis, about the nature of the errors of an instrument or a computational system, and about the nature of stability. The paper is very short, and repays reading.

Of the papers on astronomy, the one with the most mathematical interest is by Jerzy Neyman, Elizabeth L. Scott and C. D. Shane. But all the papers are of considerable astronomical interest. There are five dealing with the Hertzsprung-Russell diagram and three dealing with Spatial Distribution of Galaxies. Since the names of the authors were reported in an earlier number of this MONTHLY

(vol. 63, 1956, p. 441), they are omitted here. It should be remarked, however, that the authors are among the ablest working in these fields, and that the collection of the papers in these groupings presents an illuminating view of the subjects.

MINA REES
Hunter College, New York

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. V: *Econometrics, Industrial Research, and Psychometry*. Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956, viii+184 pp. $5.75.

This book contains ten papers presented either in December 1954 or July-August 1955 at the symposium of the title. Of the four papers in the section entitled Contributions to Econometrics, three actually are concerned with the foundations of probability and decision theory. It is to be hoped that potential readers interested in this field will discover these articles under the guise of econometrics.

The one paper in the econometrics section which does not deal with foundations is *Reduction of Constrained Maxima to Saddle-point Problems* by K. J. Arrow and L. Hurwicz. The main result here lies in two theorems which imply the existence of a local nonnegative saddle-point for a modified Lagrangian expression under conditions which are somewhat less restrictive than those assumed by Kuhn and Tucker in a similar existence theorem.

The second paper is E. W. Barankin's *Toward an Objectivistic Theory of Probability*, which sets forth "the initial ideas and implications of a . . . theory of behavior." Arguing that utility should be a set function rather than a point function, that *irrational* is to a theory of behavior as *magical* is to a theory of physics and that distinctions between conscious and unconscious motivations are vain, the author follows human free will as a *guiding light* to a mathematical characterization of the concept *personality process*. The theory yields the following: (a) *Probability* and *utility* are one and the same thing—in fact, "the single key to the full resolution of the nature of reality." (b) an individual (or any *thing*) is a complex of personality (stochastic) processes, (c) all behavior is discrete, (d) "the three concepts of *eventuality*, *probability*, and *act* express the entire substance of reality." *Probability* for the theory is given by a modified relative frequency approach which seems to avoid some of the pitfalls of von Mises' theory. It is hoped that this exciting paper will be followed by further amplification by Barankin and investigation by others.

In the third paper, *Problems of Value Measurement for Theory of Induction and Decisions*, C. W. Churchman argues that "the experimental approach to the problem of decision-making lacks a theory of data collection (*i.e.* a theory of stability of information)," and "until such a theory is at least tentatively formed, we lack any foundations for decision theory." While Barankin's theory in the

previous paper suggests possible answers to some of the questions raised by this paper, it is not clear that the theory gets at this main one.

In *The Role of Subjective Probability and Utility in Decision-Making*, Patrick Suppes takes an entirely different tack at foundations and gives an axiomatization of decision theory which is very similar to Savage's but differs from it in that (i) "the number of states of nature is arbitrary rather than infinite" and (ii) "a 50–50 randomization of two pure decisions is permitted."

In the section on industrial research A. H. Bowker surveys continuous sampling plans and suggests directions for research in this field, C. Daniel discusses fractional replication and completes the published lists of fractional replicates for the useful ranges of the number of factors and number of runs, and M. Sobel extends Birnbaum's results on a sequential procedure for selecting that particular one of two or more exponential populations with the largest expected life.

In the section on psychometry T. W. Anderson and H. Rubin discuss some methods of factor analysis. They restrict their attention to one general probability model, use it to "point up features of model-building and statistical inference that occur in other areas," and state and prove some new results for the given model. Then F. Mosteller discusses stochastic learning models for simple psychological experiments, and H. Solomon brings forth some unsolved or unclear issues for problems of item analysis and classification techniques. Each of the three papers in this section could well serve as an introduction to the topic being discussed.

In the opinion of the reviewer almost every one of the papers in this volume should serve as a stimulant to further research. The book is to be recommended to all those interested in econometrics, industrial research, psychometry, or the foundations of probability.

FRANK L. WOLF
Carleton College

Random processes in automatic control. By J. Halcombe Laning, Jr. and Richard H. Battin. McGraw-Hill, New York, 1956. ix+434 pp. $10.00.

This book is a presentation of the techniques useful in analyzing and synthesizing control systems whose inputs are random functions of time. It contains sufficient background in statistical methods to allow the control systems engineer, starting with only a speaking acquaintance with statistics, to acquire a working knowledge of these methods.

The general aim of the book is to give the engineer the tools to enable him to fashion some sort of figure of merit for a control system subject to random inputs.

Two preliminary chapters devoted to an exposition of *Basic Concepts of Probability Theory* and *Descriptions of Random Processes* prepare the student (or engineer) for the analyses of actual systems. The properties of correlation functions, stationary and ergodic processes, and the energy spectral density are all

treated. These chapters are interlarded with enough specific examples, illustrative of the ideas presented in the text, that the reader can gain some facility in giving analytic expression to physical statistical problems. These preliminary chapters are sufficiently comprehensive that the reader need not, in the subsequent discussions, be limited by only an intuitive grasp of probability and random processes. On the other hand, a complete assimilation of this background material is not required in order to make use of the techniques presented in the principal chapters concerned with systems engineering.

In these subsequent chapters the extensive use of the mean-squared error as a performance index and its limitations are discussed candidly. In fact, the careful delineation of the state of the techniques discussed is one of the principal features of the book. For those areas for which inadequate analytical theory exists, the situation is outlined and the analogue computation methods, which take the place of the undeveloped mathematical theory, are explained. Since this is the situation for systems with variable coefficients, or with non-stationary inputs, these analogue computing methods are finding much use in analyzing actual control system problems such as occur in fire control system design. The time-saving method of adjoint simulation for obtaining mean-squared errors is spelled out in some detail. For systems with stationary inputs, and constant coefficient control systems, mathematical means are developed for obtaining the mean-squared error in terms of the system weighting function, or transfer function, and the input spectral density.

The final two chapters in the book deal with system synthesis rather than analysis. The Wiener theory for synthesizing the optimum system weighting function is developed. In the last chapter it is extended to cases for which the input signal is not stationary, and for which the past history is known over only a limited time.

A well annotated bibliography is provided, and a short but complete explanation of the analogue computer is given in an appendix.

For either the student or the practising control systems engineer, this book offers an understandable statement of the current techniques (and their limitations) available for the analysis and synthesis of control systems subject to random input signals.

W. F. CARTWRIGHT
U. S. Naval Ordnance Test Station
China Lake, California

Topics in Number Theory. By William J. LeVeque. Addison-Wesley, Reading, 1956. vol. I, x+198 pp. $5.50; vol. II, viii+270 pp. $6.50.

These two volumes constitute an outstanding contribution to the literature of number theory, because there is much material in volume II, and also in the later chapters of volume I, which is not readily accessible elsewhere. The author develops the subject from the beginning, presuming no previous knowledge of

number theory on the part of the reader. Although a wide variety of topics is presented in the two volumes, the writer has placed much more emphasis on analytic number theory, and less emphasis on Diophantine equations and quadratic forms, than most American books on the subject.

Volume I is suitable for a first course for advanced undergraduate and beginning graduate students. Although calculus is used in two of the nine chapters, no other technical knowledge is required. The writing, while very clear and in some places remarkably illuminating, is at a fairly sophisticated level and will not be easy going for the mediocre student. Similarly, while there is a good collection of problems at the ends of the sections, very few of them are routine computational exercises. Clearly the author has had in mind an audience of serious, competent students of mathematics.

Volume I opens with a chapter on the nature of the subject and the types of proof used in mathematics. This is followed by several chapters comprising the basis of number theory: the Euclidean algorithm, congruences, linear Diophantine equations, primitive roots and quadratic residues. The writer then turns to number-theoretic functions and applications to elementary results in the theory of prime numbers, for example the Erdös proof of the Bertrand conjecture. The representation of a number as a sum of two squares, and the fact that every positive integer is a sum of four squares, are given. In Chapter 8 there is a treatment of the Pell equation with various applications, and also a proof, based on Farey series, of the Hurwitz theorem that to any irrational number ξ there correspond infinitely many rational numbers x/y such that $|\xi - x/y| < 1/(\sqrt{5}y^2)$. This proof is not based on continued fractions. In fact, continued fractions are introduced in the last chapter of volume I in a rather novel way: as a device to get the "good" rational approximations to a real number.

One of the author's main reasons for writing the books was to provide the advanced topics of volume II. Here the level of mathematical maturity required for understanding is much higher, a working knowledge of the theory of analytic functions being a prerequisite in two of the seven chapters. The first chapter treats binary quadratic forms from the geometric viewpoint based on the modular group. This formulation provides an interesting contrast with the arithmetic treatment commonly given. The next two chapters are concerned with algebraic numbers and applications to rational number theory, principally Kummer's theorem that for a regular prime p the equation $x^p + y^p + z^p = 0$ has no solution in rational integers x, y, z for which $(p, xyz) = 1$, and the Delaunay-Nagell theorem that $x^3 + dy^3 = 1$ has at most one non-trivial solution for any given integer d.

Chapter 4 is titled *The Thue-Siegel-Roth Theorem*, and it could have been labeled *The Liouville-Thue-Siegel-Dyson-Schneider-Roth-Le Veque Theorem*. Without citing all the results in this chain, let us recall that Liouville proved that if α is any algebraic number of degree $n \geqq 2$, then there is only a finite number of rationals h/k satisfying the inequality $|\alpha - h/k| < 1/(k^t)$ for fixed real $t > n$. (By use of this result, Liouville was the first to establish the existence of transcendental numbers.) Roth proved in 1955 that there is only a finite number of

rationals h/k satisfying the inequality for fixed $t>2$. LeVeque extends this to an algebraic formulation (for the first time in print, apparently) as follows. For any algebraic number β define $H(\beta)$ as the maximum of the absolute values of the coefficients of the unique monic minimal polynomial of β. Let K be an algebraic number field, and let α be algebraic of degree $n \geqq 2$ over K. Then for fixed real $t>2$ there is only a finite number of solutions β in K satisfying $|\alpha-\beta| < 1/(H(\beta))^t$.

Irrationality and Transcendence is the title of Chapter 5, which includes recent work of Mahler on arithmetic properties of the exponential function, of which the generalized Lindemann theorem and the transcendence of π are corollaries. Also given is a general formulation by Schneider of the Gelfond result that a^b is transcendental (apart from obvious special cases) for algebraic a and b.

The last two chapters deal with the distribution of prime numbers. Chapter 6 includes two proofs of the Dirichlet theorem on the infinitude of primes in an arithmetic progression, one non-elementary and one elementary, *i.e.* with and without the use of analytic function theory. The prime number theorem is proved (by non-elementary methods) in the final chapter, and an extension is made to the asymptotic estimate of the number of primes in an arithmetic progression.

There are notes at the ends of the chapters disclosing much of the author's source material, which will be most helpful to every student of number theory. Furthermore, the books are enlivened by frequent discussions of the author's reasons for choosing one approach rather than another, one proof rather than another. These handsomely printed volumes provide a welcome addition to the advanced expository literature.

IVAN NIVEN
University of Oregon

Applied Analysis. By C. Lanczos. Prentice-Hall, Englewood Cliffs, N. J., 1957. xx+539 pp. $9.00.

The book under review is designed to serve as an introduction to one of the most vital and significant fields of current mathematical research, the field of computational analysis. It serves its purpose admirably.

The style is pleasantly informal and leisurely, yet an enormous amount of material is included. In each chapter there is an interesting historical introduction to the principal problems to be discussed, followed by clear-cut formulations of these problems, a description of the methods to be employed in their treatment, and a number of completely worked-through numerical examples.

Much of the material is of current vintage, developed by the author himself in the course of many years spent in the company of digital computers. Occasionally the author cannot resist the temptation to pluck a plum from higher mathematical branches than those with which the remainder of the book deals. However, this is not necessarily a fault and should serve to whet the appetite of the reader for further reading in the field of analysis.

Let us now briefly survey the contents of the various chapters.

I. *Algebraic equations.* Since determination of the roots of a polynomial equation plays an essential part in stability analysis, a number of the most frequently used methods are presented in this opening chapter.

II, III. *Matrices and eigenvalue problems.* Matrices are introduced in conjunction with the eigenvalue problem and its geometric interpretation. A number of iterative techniques are discussed in connection with the problem of solving systems of linear equations of large degree.

IV. *Harmonic analysis.* The author motivates the treatment of finite Fourier series by means of an interpolation problem for data collected at equidistant time intervals. From there he goes on to the subject of Fourier series and Fourier integrals, and continues with a cross-section of the theory of orthogonal expansions.

Particularly useful is his discussion of the numerical inversion problem for the Laplace transform which escapes the tables. This is seldom discussed.

V. *Data processing.* This chapter contains a discussion of a number of important topics arising in the use of empirical functions defined by means of numerical tables. There is a discussion of least squares, determination of derivatives of empirical functions, and other aspects of the general smoothing problem.

VI. *Quadrature methods.* Here the problem is that of evaluating definite integrals of functions which may be empirical, or which may not possess simple analytic integrals. In addition to the classical methods of Simpson and Gauss, a number of other techniques are discussed as well.

VII. *Power expansions.* Although the method of power series expansions is one of the most powerful in analysis, it cannot be applied efficiently in the majority of cases without careful planning. The author discusses a number of ways in which the convergence of these expansions may be greatly increased with particular emphasis upon the use of Chebyshev polynomials.

An important application of these techniques is to the solution of differential equations with rational coefficients.

Finally let us note that the book is printed in a very readable and attractive type.

RICHARD BELLMAN
The RAND Corporation

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor W. L. Williams, University of South Carolina, was the representative of the Association at the inauguration of President F. R. Veal of Allen University on March 30, 1957.

Institute of Mathematical Sciences, New York University: Professor William Eberlein, University of Wisconsin, Dr. Joe Foote, University of Oklahoma, Dr. Frank Karal, Jr., Magnolia Petroleum Company, Dr. Martin Kneser, University of Heidelberg, Dr. Paul Koosis, University of Michigan, Dr. Walter Kyner, Northwestern University, Mr. Seymour Parter, Los Alamos Scientific Laboratory, Professor H. E. Rauch, University of Pennsylvania, Dr. Gian-Carlo Rota, Yale University, Professor J. T. Schwartz, Yale University, Dr. Soloman Schwartzman, Johns Hopkins University, and Dr. Shlomo Sternberg, Johns Hopkins University are in residence under the Temporary Membership Program.

Northwestern University: Visiting Associate Professor H. C. Wang, Columbia University, has been appointed Associate Professor; Dr. Teruhisa Matsusaka, University of Chicago, has been appointed Visiting Associate Professor.

University of Kentucky: Associate Professor Sallie E. Pence has been promoted to Professor; Mr. Richard Sprague, part-time instructor, has been appointed Instructor; Professor V. F. Cowling has been made Research Professor under a grant from the National Science Foundation; Mr. J. B. Cornelison and Mr. J. B. Wells have been made research instructors and are working on the National Science Foundation Contract project.

Vassar College: Associate Professor Abba V. Newton has been promoted to Professor; Associate Professor Winifred Asprey has been awarded an IBM Post-Doctoral Industrial-Research Fellowship for 1957–58 and will be on leave of absence.

Associate Professor J. W. Andrushkiw, Seton Hall University, has been promoted to Professor.

Dr. J. H. Bell, Head, Radar Development Department, Research Laboratory Division, Bendix Aviation Corporation, Detroit, Michigan, has a position as a scientific staff consultant at AC Spark Plug Division, General Motors Corporation, Milwaukee, Wisconsin.

Dr. H. F. Bright, Deputy Director, Technical Services, Human Resources Research Office, George Washington University, is now an operations research specialist at General Electric Company, Plainville, Connecticut.

Mr. C. B. Brown is a mathematician at Sohio Petroleum Company, Oklahoma City, Oklahoma.

Assistant Professor P. L. Butzer, McGill University, is on leave of absence and has been appointed Visiting Professor at the University of Mainz, Germany.

Mr. D. A. Cope, Research Engineer, Consolidated-Vultee Aircraft Corporation, San Diego, California, is a project engineer for Technical Research Group, New York, New York.

Mr. H. L. Dachslager, has been appointed a mathematician-economist at Remington Rand Univac, St. Paul, Minnesota.

Dr. Gus DiAntonio, Carnegie Institute of Technology, has accepted a position as aerodynamicist with Bell Aircraft Corporation, Buffalo, New York.

Mr. E. L. Dubowsky, Teacher, Colby Community High School, Kansas, has been appointed Instructor at the University of Wichita.

Associate Professor Charles Fox, McGill University, has been promoted to Professor.

Mr. C. B. Germain, Graduate Student, Iowa State College, has been appointed Assistant Professor of Statistics at the University of Manitoba.

Mr. T. L. Glahn, Technical Engineer, General Electric Company, Evendale, Ohio, has a position as a design specialist at Glenn L. Martin Company, Denver, Colorado.

Assistant Professor Simon Green, University of Tulsa, has been promoted to Associate Professor.

Assistant Professor Gerald Harrison, Wayne State University, has accepted a position as a systems engineer with Teleregister Corporation, Stamford, Connecticut.

Dr. J. F. Heyda, General Motors Corporation, has accepted a position as a mathematician with General Electric Company, Cincinnati, Ohio.

Mr. J. F. Jakobsen, Instructor, University of Missouri, has been appointed Instructor at the University of New Hampshire.

Mr. J. B. Lackey, University of Kentucky, has been appointed a mathematical analyst at Ballistics Reduction Group, Ordnance Test Activity, Yuma Test Station, Arizona.

Mr. G. M. Leibowitz has a position as an assistant engineer at the Ford Instrument Company, Missiles Development Division, Long Island City, New York.

Professor C. B. Lindquist, University of Minnesota, Duluth, has been appointed Chief for Natural Sciences and Mathematics, Department of Health, Education, and Welfare, Office of Education, Washington, D. C.

Assistant Professor D. S. McManus, Norwich University, has been appointed Instructor at the University of Wisconsin.

Dr. G. W. Medlin, Wake Forest College, is on leave of absence and is a mathematician at Oak Ridge National Laboratory, Tennessee.

Assistant Professor W. K. Moore, Albion College, has been promoted to Associate Professor.

Mr. J. E. Mullins, Mathematician, North American Aviation, Propulsion Field Laboratory, Santa Susana, California, is now an assistant mathematician at the Rand Corporation, Lincoln Laboratory, Lexington, Massachusetts.

Mr. F. E. Nemmers has a position as an analyst with the AC Spark Plug Division, General Motors Corporation, Milwaukee, Wisconsin.

Mr. E. F. Ormsby has been appointed Director of EDPM, International Business Machines World Trade Corporation, New York, New York.

Assistant Professor W. A. Rutledge, University of Tulsa, has been promoted to Associate Professor.

Miss Inez I. Sausen, Instructor, Garden City Junior College, has been appointed Instructor at the University of Wichita.

Dr. Paul Slepian, Ramo-Wooldridge Corporation, has a position as a research physicist, Hughes Research and Development Laboratories, Culver City, California.

Dr. M. D. Springer, Naval Ordnance Plant, Indianapolis, has accepted a position as a senior operations analyst with Technical Operations, Fort Monroe, Virginia.

Associate Professor D. D. Strebe, University of South Carolina, has been appointed Professor at the Teachers College at Oswego, State University of New York, effective September 1957.

Mr. Ralph Surasky, Graduate Assistant, University of South Carolina, has been appointed Assistant Professor at North Georgia College.

Dr. F. B. Thompson, Project Engineer, General Analysis Corporation, Santa Monica,

California, is now Manager, Operations Analysis, Computer Department, General Electric Company, Phoenix, Arizona.

Mr. A. W. Wallace, Instructor, University of Massachusetts, has a position as a mathematician at Instrumentation Laboratory, Massachusetts Institute of Technology.

Miss Marjorie Watson, Chattanooga, Tennessee, has accepted a position as engineer with Westinghouse Electric Corporation, Baltimore, Maryland.

Dr. G. L. Weiss has been appointed to an instructorship at DePaul University.

Mr. Nathan Wetrogan, Mathematical Engineer, Lockheed Aircraft Corporation, Marietta, Georgia, has a position as a research engineer at Grumman Aircraft Engineering Corporation, New York, New York.

Dr. W. D. Wood, Sandia Corporation, Albuquerque, New Mexico, has accepted a position as Vice-President of the Dikewood Corporation, Albuquerque.

Professor H. E. Arnold, Wesleyan University, died on March 2, 1957. He was a member of the Association for thirty-three years.

Mr. J. H. Fishel, Research Assistant, University of Illinois, died on January 20, 1957.

Professor N. A. Goldsmith, Chicago Teachers College, died on January 20, 1957. He was a member of the Association for seven years.

Professor Emeritus J. M. Howie, Nebraska Wesleyan University, died on December 7, 1956. He was a charter member of the Association.

Rev. A. J. O'Leary, Chairman, Department of Mathematics, St. Anselm's College, died on February 3, 1957. He was a member of the Association for twenty-three years.

Professor J. M. Rankin, College of Idaho, died on June 8, 1956. He was a member of the Association for thirty-nine years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 117 persons have been elected to membership by the Board of Governors on applications duly certified.

WALTER L. ANDERSON, M.S. (West Virginia) Instr., West Virginia University.

MILTON T. AUSTIN, Student, Rutgers University.

RAY AUTHEMENT, Ph.D. (Louisiana S.U.) Asso. Professor, McNeese State College.

MRS. ANNE B. BARNES, M.A. (Texas) Instr., University of Texas.

BETTY L. BERNHARDT, M.A. (Ohio S.U.) Instr., Ohio University.

ROBERT T. BLACKBURN, Ph.D. (Chicago) Asst. Professor, San Francisco State College.

JOAN B. BOENITSCH, A.B. (Mary Washington) Math., Army Map Service, Washington, D. C.

JOSEPH R. BRASHEAR, Student, Carnegie Institute of Technology.

MRS. HELEN H. BROSS, Ph.D. (Yale) Asst. Professor, Talladega College.

WALTER C. BUTLER, M.A. (Colorado S.C.) Asst. Professor, Colorado Agricultural and Mechanical College.

WILLIE B. CAMPBELL, M.A. (Columbia) Instr., Texas Southern University.

MR. CHU C. CHANG, M.A. (Washington) Instr., Seattle University.

BENJAMIN CHEREEK, M.S. (Georgia) Analyst,

Air Force Armament Center, Eglin Air Force Base, Fla.

WILLIAM H. COLBERT, JR., B.S. (Nevada) Grad. Asst., University of Nevada.

CAROL A. CONN, M.A. (Cornell) Instr., Hobart College.

DONALD CONWAY, A.B. (San Jose S.C.) Instr., San Jose Junior College.

GEORGE T. CROCKER, B.S. (Union U.) Teaching Fellow, Alabama Polytechnic Institute.

MORTON L. CURTIS, Ph.D. (Michigan) Asso. Professor, University of Georgia.

MERLE L. DECKARD, Student, Lawrence Institute of Technology; Product Designer, Ford Motor Co., Detroit, Mich.

GUS DIANTONIO, Ph.D. (Pittsburgh) Aerophysicist, Bell Aircraft Corp., Niagara Falls, N. Y.

MORTON M. DICKSTEIN, B.A. (Hunter) Actuarial Trainee, George B. Buck & Co., New York, N. Y.

WILLIAM DONAHUE, U. S. Navy, Retired, Long Beach, California.

C. TOLAND DRAPER, B.S. (Lane) Grad. Student, University of California, Berkeley.

DONALD S. EVANS, B.S. (Stanford) Project Engr., Electro Instruments, San Diego, Calif.

LOUIS J. EVANS, B.S. (Detroit) Stress Analyst, Long Manufacturing Co., Detroit, Mich.

RUDOLF N. FESTA, Ph.D. (Vienna) Visiting Professor, University of Alabama.

FLOYD G. FISHER, Ph.D. (California) Math., Bureau of Ordnance, Dept. of Navy.

HAROLD FLEISHER, Ph.D. (Case I.T.) Manager, Research Communications, I.B.M., Poughkeepsie, N. Y.

EDWARD H. FLETCHER, B.A. (California, Berkeley) Programming Math., National Advisory Committee on Aeronautics, Cleveland, Ohio.

EUGENE A. FRANCIS, M.A. (Columbia) Asst. Professor, University of Puerto Rico.

MR. AUBYN FREED, Ph.D. (Illinois) Instr., Smith College.

MARION R. FRIEL, A.M. (Loyola) Teacher, Wright Junior College.

JOHN H. GLOVER, A.B. (Michigan); B.S. (M.I.T.) Product Design Engr., Ford Motor Co., Dearborn, Michigan.

HERMAN R. GLUCK, Student, New York University.

JOHN L. GORDON, B.S. (Texas Tech.C.) Math., Holloman Air Development Center, N. Mex.

FEDERICO GRABIEL, M.S. (Chicago) Res. Physicist, Hughes Research and Development Labs., Culver City, Calif.

JOHN W. GREINER, M.A. (Colgate) Asst. Professor, University of Florida.

ROLLIE J. HARP, M.S. (Florida S.U.) Instr., University of Chattanooga.

JAMES L. HATFIELD, M.A. (Virginia) Asst., Professor, College of William and Mary, Norfolk.

MICHAEL HERSCHORN, M.A. (McGill) Lecturer, McGill University.

ABRAHAM P. HILLMAN, Ph.D. (Princeton) Asst. Professor, State College of Washington.

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TRAVEL GRANTS FOR THE 1958 INTERNATIONAL CONGRESS OF MATHEMATICIANS

Travel grants will be made to a limited number of mathematicians who wish to attend the International Congress of Mathematicians in Edinburgh, August 14–21, 1958. It is expected that funds available may provide travel assistance for about twenty-five mathematicians. Grants will be made on the basis of recommendations by the Committee on Travel Grants, which has been appointed by the Division of Mathematics of the National Academy of Sciences—National Research Council. In order to work out a fair distribution, the Committee has asked for the cooperation of various mathematical societies. A committee of the Mathematical Association of America will submit to the Committee on Travel Grants an ordered list of those members of the Association who desire grants.

Members of the Association who desire travel grants should (1) notify the Secretary of the Association, Professor H. M. Gehman, University of Buffalo, Buffalo 14, N. Y., and supply the information described below, (2) send a written request to the National Science Foundation, Washington 25, D. C., for an application form for foreign travel grants, and (3) fill out this form and return it to the National Science Foundation. The deadline for receiving applications is October 15.

The additional information referred to above, to be sent to the Secretary of the Association, is as follows: Each applicant should (1) send his complete address, (2) state whether he intends to present a paper and, if so, whether this is by invitation, and (3) describe any other travel funds that are available to him. Note that each application to the National Science Foundation should be matched by an application through the Association as described above.

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-fourth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi on February 15–16, 1957. The Friday afternoon meeting was held in two concurrent sessions. Professor Elsie T. Church, Louisiana Vice-Chairman and Professor Roy D. Sheffield, Mississippi Vice-Chairman presided. Dean W. H. Bradford, McNeese State College, Chairman of the Section, presided at the Friday evening and Saturday morning sessions.

There were 117 persons registered including 65 members of the Association.

The following officers were elected for the coming year: Chairman, Professor A. C. Grimes, Mississippi State College; Vice-Chairman for Louisiana, Professor S. M. Spencer, Louisiana College; Vice-Chairman for Mississippi, Professor N. A. Childress, University of Mississippi; Secretary-Treasurer, Professor T. L. Reynolds, Millsaps College.

At the business meeting the section approved the sponsoring of a high school mathematics contest jointly with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics. A 15 member committee was appointed by the chairman with a 5 man steering committee, Professor H. T. Karnes of Louisiana State University as chairman. A full report of this committee is to be brought before the section at the next meeting. A membership committee for the section was also appointed.

The invited speaker for the meeting was Dr. William T. Guy, Jr. of the University of Texas. His lecture on Friday evening was entitled "What Do Mathematicians Do?" He discussed some weaknesses of mathematicians and gave some inspirational challenges on how to make mathematics alive. The Saturday morning address was on "Cybernetics." The relation of cybernetics to statistical mechanics, communication and control was discussed.

The following papers were presented:

1. *The new look in mathematics*, by Professor Lester M. Garrison, Louisiana Polytechnic Institute.

This talk summarized some of the recent developments in high school and college mathematics and the new emphasis on modern mathematics. The need for special programs for the gifted students and the recommendations of the Committee on Study of Admission with Advanced Standing and the M.A.A. Committee on the Undergraduate Program were discussed.

2. *Making better use of the etymology of mathematical terms as an aid in teaching*, by Professor T. F. Mulcrone, S. J., Loyola University.

The speaker pointed out that the following objectives are better attained by an interest in the etymology of mathematical terms: (1) a better understanding of the mathematics taught, (2) the remembering of definitions, (3) the identification of concepts in their historical setting, and (4) the generalization and unification of mathematical types. Some methods were suggested for introducing in the class room references to the root meaning of words.

3. *Certain constructions completed*, by Professor Benjamin Ernest Mitchell, University of Mississippi.

If we are to have free and unrestricted use of the terms "continuity," "conversely," "in general" and "without exception" in our geometry, we must come to grips with the imaginary. It cannot be dumped into the discard, nor shunted to the side track, nor carried as excess baggage. It must be part and parcel of the paraphernalia of geometry. The explication at such a place for the imaginary is the objective of this paper as it relates to certain constructions. In particular the constructions of triangles in the cases $s-s-s$ and $s-s-a$, and the common harmonic conjugate of two given point-pairs on a line or two line-pairs on a point.

4. *Squares and diamonds*, by Professor V. B. Temple, Mississippi Southern College.

Draw circles C_1: $\rho_1 = 2a_1 \cos\theta$ and C_2: $\rho_2 = 2a_2 \sin\theta$. From the origin draw any line cutting C_1 in H_1 and C_2 in H_2. A horizontal through H_2 and a vertical through H_1 meet in P_1; a vertical through H_2 and a horizontal through H_1 meet in P_2. P_1 and P_2 define the loci of two continuous and closed line segments perpendicular to each other and whose equations are:

$$P_1: \begin{cases} x = 2a_1 \cos^2\theta, \\ y = 2a_2 \sin^2\theta. \end{cases} \qquad P_2: \begin{cases} x = 2a_2 \sin\theta \cos\theta, \\ y = 2a_1 \sin\theta \cos\theta. \end{cases}$$

If $a_1 = a_2$ we have the T square; if $a_1 \neq a_2$ the L squares. If double signs are used in the equations they are those of a square when $a_1 = a_2$, and of a diamond when $a_1 \neq a_2$.

5. *Curve tracing*, by Professor Elsie T. Church, Northwestern State College.

The paper presented a brief description of the use of the analytical triangle and the analytical polygon to find the approximate form of a curve at certain points, to determine all of the asymptotes and to determine the intercepts on the X-axis, Y-axis and the line at infinity. Several examples were given.

6. *The STIP program*, by Professor Houston T. Karnes, Louisiana State University.

A survey of the Science Teaching Improvement Program was given. This included the conditions which gave rise to the program and an outline of the work to date. The paper ended with a discussion of the work of the Regional Consultants which is the most recent phase of the program. The author is one of these consultants.

7. *The graduate program in mathematics at Louisiana State University*, by Professor R. D. Anderson, Louisiana State University.

A survey is given of the various facets of the graduate program in mathematics at Louisiana State University. Topics covered include the mathematical fields of interests of the members of the faculty; the available means of support for graduate students; course offerings, seminars and colloquia; and plans for future activity.

8. *A demonstration on the use of the electric analog computer*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

The differential equation $(D^2+2.4D+144)y=250$ is solved. The equation is first transformed into the machine equation by proper scale factors. The machine was brought along and several demonstrations were given during the meeting.

9. *A general theory for linear systems*, by Professor Roy D. Sheffield, University of Mississippi and Convair, Inc.

For a singular matrix L, a pseudo-inverse M of L is defined by the relation $LML=L$. If a matrix equation $Lx=b$ is consistent, then it follows that Mb is a particular solution; moreover, the column vectors of $I-ML$ span the linear space of solutions of $Lx=0$. If L is an $n\times n$ matrix of rank r, then a calculation for an M is given whereby $I-ML$ has exactly $n-r$ non-zero columns. In particular, this means that a system of linear equations can be solved by machine methods regardless of the nature of the matrix.

10. *Another note on quasi-idempotent matrices*, by Professor B. E. Mitchell, Louisiana State University.

The canonical form of a quasi-idempotent matrix is determined. Certain properties of quasi-idempotent matrices are proved by means of this canonical form.

11. *Centrifugal multiplication*, by Professor Haskell Cohen, Louisiana State University, introduced by the Secretary.

R. J. Koch has asked whether there exists a topological semigroup S, defined on the two cell with the property that $S^2 = \{xy: x \text{ and } y \in S\}$ contains the boundary of S but $S^3 \neq S$. The example given below not only meets the desired conditions but actually has S^3 equal to the boundary of S. Let S be the unit disk in the complex plane with center at the origin. Using polar coordinates, define the product of (r_1, θ_1) and (r_2, θ_2) to be the point $(1, 2\pi \max\{r_1, r_2\})$. It is easy to check the required properties.

T. L. REYNOLDS, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 15–16, 1957, at Emory University, Emory University, Georgia. Professor G. B. Huff, Chairman of the Section, and Professor Trevor Evans presided over the general sessions; Professors R. G. Blake, C. G. Latimer, J. D. Novak, W. L. Strother and M. C. Wicht presided over subsections.

There were about 300 in attendance including 185 members of the Association.

The following officers were elected for the coming year: Chairman, Professor Trevor Evans, Emory University; Vice-Chairman, Professor D. E. South, University of Florida; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College. The following were named as the Section's Committee on High School Contests, and will report at the next annual meeting: Professors J. B. Banks (Chairman), A. H. Herrington, E. Myrtice Lynch, J. D. Mancill, E. D. Nichols, J. W. Sawyer, C. Eucebia Shuler and F. L. Wren.

The following program was presented:

1. *Numbers that are irrational*, by Professor C. G. Phipps, University of Florida.

An extension of the note in this MONTHLY, vol. 63, 1956, p. 247, to other bases of notation leads easily to conclusions about the irrationality of the various roots of 2 and other positive integers.

2. *Some test questions in mathematics: A comparison between the Union of Soviet Socialist Republic and the United States*, by Professor Robert Kalin, Florida State University.

To enroll as freshmen in the Mechanics-Mathematics Department at the University of Moscow, U.S.S.R., applicants must pass an examination consisting of three algebra and three geometry problems. Thirty-five sample problems from the 1950–53 examinations indicate that greater skill in following a maze of algebraic technique and geometric information is required than is needed to handle the *typical* U. S. college entrance test. However, the University of Moscow may be *atypical*, and some U. S. examinations require a higher level of mathematical attainment and perhaps greater understanding of basic mathematical principles.

3. *Mathematics for social scientists*, by Professor E. B. Shanks, Vanderbilt University.

An experimental course at Vanderbilt for graduate students in the social sciences, based upon recommendations of the Madow Committee, was discussed. The point of view that many essential concepts in mathematics could be "freed" from mathematical contexts and thereby more easily applied to other fields of knowledge was expressed and illustrated; also, that constructive definitions should replace descriptive, where feasible. As an illustration, groups were discussed as certain types of square arrays. An important feature of the latter discussion was the consideration of the associative law stated in the following manner: $ab = c$ implies $a(bx) = cx$.

4. *Āryabhaṭa, first among the mathematicians of India*, by Mr. W. J. Mays, Actuary Imperial Life Insurance Company.

Āryabhaṭa (476 A.D.—c. 550 A.D.) was the most celebrated among a very early group of Indian astronomer-mathematicians. The Sanskrit treatise known as the *Āryabhaṭīya* (edited by Dr. H. Kern, Leiden, 1874) is attributed to him. It contains 118 aphorisms, or maxims, on astronomy and mathematics, composed in verse. Of particular interest are Āryabhaṭa's treatments of quadratic equations and linear indeterminate equations, as well as his famous approximation of *pi*, 3.1416. Although no proof or derivation of the results is given, a critical study of the text leads to certain plausible conjectures about some of his reasoning.

5. *New amicable pairs*, by Professor Mariano Garcia, Jr., University of Puerto Rico.

This paper lists over one hundred and fifty new amicable pairs obtained by methods related to those used for finding multiply perfect numbers. Several of the new pairs are of the "miscellaneous" type, of which relatively few examples are known.

6. *Properties of quasi-idempotent matrices*, by Professor G. B. Huff, University of Georgia.

A square matrix $A=(a_{ij})$ over the field of complex numbers is said to be *quasi-idempotent* if there is a polynomial matrix $(f_{ij}(x))$ such that for each positive integer r, $A^r=(f_{ij}(r))$. A matrix A is quasi-idempotent if there is a positive integer k such that $(A-E)^kA=0$, where E is the identity matrix; and, in this case, $f_{ij}(x)$ is of virtual degree $(k-1)$ for all i, j. (This MONTHLY, Vol. 62, 1955, pp. 334–339). Professor Huff pointed out the role of these matrices in elementary matrix theory and indicated some applications.

7. *On the universal sums of nine values of a cubic function*, by Professor R. L. Yates, University of Florida.

Necessary conditions are found for the coefficients ϵ, σ, ρ in order that the function $F(x)=(\epsilon/6)(x^3-x)+\sigma x+\rho$ may have nonnegative integral values for all the integers $x\geqq t$ and represent 0 and 1 for two such values. This gives rise to formulae for determining all such possible functions for which all integers are represented as the sum of nine values.

8. *A note on the Laguerre quadratic formula for the solution of algebraic equations*, by Professor Stephen Kulik, University of South Carolina.

Let $f(x)=0$ be an algebraic equation of degree N with real roots. Let x be a real number between any two roots of the equation, u an arbitrary number not equal to x or a root of the equation, and n an even number. Then the two real values of x_1, which satisfy the equation $[Q_n(u,x)+Nf^n(x)](x-x_1)^n=(u-x_1)^nf^n(x)$, where $Q_n(u,x)$ depends on $f(x)$ and its first n derivatives and is calculated recursively, approximate both roots adjacent to x. The accuracy increases with the increase of n.

9. *On a universal ternary quadratic form*, by Professor E. H. Hadlock, University of Florida.

Let C_1, C_2 be any given positive, odd and relatively prime integers. Let C_3 satisfy the congruences $C_3\equiv -C_2 \pmod{C_1}$, $C_3\equiv -C_1 \pmod{C_2}$ and its prime factors satisfy certain congruences mod $8C_1C_2$. Let $C_4=C_1+C_2-C_3$ and C_3 be greater than C_1+C_2 if $C_3>0$. If the prime factors of C_4 also satisfy certain congruences mod $8C_1C_2$, then the form $f=2X^2+C_1C_2Y^2+C_3C_4Z^2$ is universal. In particular if C_1, C_2 and C_3 are restricted to classes modulo 8, then there exist infinitely many primes $|C_4|=C_1+C_2-C_3$ for which f is universal.

10. *Examples of maximal non-prime ideals*, by Professor M. L. Curtis, University of Georgia.

In teaching modern algebra one proves that in a commutative ring with unit every maximal ideal is prime. There are very simple examples to show that both commutativity and a unit are necessary. This note gives a presentation which yields a class of such examples and has side results which are interesting from a pedagogical point of view.

11. *On the solutions of a binary quadratic congruence*, by Professor C. G. Latimer, Emory University.

In this paper one considers the binary congruence $x^2+y^2+1\equiv 0 \pmod{m}$, where m is a positive odd integer. The function Ψ is defined by

$$\Psi(m) = m[1 - p_1^{-1}(-1 \mid p_1)][1 - p_2^{-1}(-1 \mid p_2)] \cdots [1 - p_n^{-1}(-1 \mid p_n)]$$

where the p's are the distinct prime factors of m and $(-1 \mid p)$ is Legendre's Symbol. It is shown that the number of distinct solutions is $\Psi(m)$ where the solutions (x_1, y_1), (x_2, y_2) are considered as distinct only if $x_1 \not\equiv x_2$ or $y_1 \not\equiv y_2 \pmod{m}$. Note that x^2+y^2 is the norm of a Gaussian complex integer. The proof uses properties of such integers and may be extended to the integers in any quadratic field.

12. *Veblen-Wedderburn systems*, by Professor J. R. Wesson, Birmingham Southern College.

In 1907, Veblen and Wedderburn showed that certain finite algebras could be used to coordinatize some finite projective planes. There are redundancies in the postulates, some more apparent than others. A unit may be introduced by modifying the system slightly, and the new systems are usually employed in the construction of "Veblen-Wedderburn" planes. The Veblen-Wedderburn system with eight elements is a field.

13. *A vector proof of Euler's theorem on rotations*, by Professor M. K. Fort, Jr., University of Georgia.

Using elementary properties of vectors (especially properties of the triple scalar product), it is shown that if T is an orientation-preserving isometry of 3-space and 0 is a point such that $T(0)=0$, then there exists $P\neq 0$ such that $T(P)=P$. Euler's theorem on rotations follows easily from this fact. The proof given differs from the usual one in that no use is made of the theory of determinants and linear equations.

14. *Planar families of lines*, by Professor Andrew Sobczyk, University of Florida.

A family of lines $F=\{-x \sin\alpha+y\cos\alpha=d(\alpha)\}$ in the Euclidean plane E_2 is *representative* if it contains exactly one line in each direction α, $0\leqq\alpha<\pi$, *continuous* if the function $d(\alpha)$, extended by $d(\pi)=-d(0)$, is continuous on $0\leqq\alpha\leqq\pi$. A point P is *simply covered* if there is exactly one line L in F which passes through P. The following theorems are proved. Each representative and continuous planar family F fills the plane. Each planar family of lines such that the points exterior to a circle are simply covered, which is not a sheaf of parallel lines, must be representative and continuous.

15. *The Apollonius contact problem of three circles*, by Professor C. N. Mills, Florida State University.

Historically the circle-contact problem was first considered by Archimedes, but it is generally known as the problem of Apollonius. This paper considers the ten cases of geometrical construction from three null circles, to the general problem of three circles tangent externally mutually two by two. The constructions make use of many modern geometry theorems. When the three circles are mutually external, but not tangent, no general algebraic solution has been given which will give the radius of each of the eight circles. Fermat was first to give a solution of the problem to determine the radii of the two spheres tangent to four spheres tangent externally three by three.

16. *Generalizations of curvature properties of curves on a surface*, by Professor J. D. Novak, University of South Carolina.

At a point P of a curve on a surface, the T-normal curvature, the T-geodesic curvature, and the T-geodesic torsion of the curve relative to a variable tangent vector of the surface are defined. Certain generalizations are obtained which reduce to the classical theorems when the variable vector

is tangent to the curve. Analogues of Euler's equation and the Dupin indicatrix are proved for the principal T-normal curvatures at P.

17. *Remarks on points of uniform convergence*, by Professor C. W. McArthur, Florida State University.

If the sequence of functions (f_n) and the function f are defined on a topological space X to a metric space Y let $B=[x_0 \in X : \epsilon > 0$ implies that there is an integer N and an open set G about x_0, such that $d(f_n(x), f(x)) < \epsilon$ if $n > N$ and $x \in G]$. Let A be the set of x_0 satisfying the above with $f(x)$ replaced by $f(x_0)$. In general, neither set is necessarily a subset of the other. If $\lim_{n\to\infty} f_n(x) = f(x)$ on X then $A \subset B$ and f is continuous at each $x \in A$. If, in addition, f_n is continuous for infinitely many n, $A = B$.

18. *On a problem of Henriksen and Isbell*, by Professor J. D. McKnight, Jr., University of South Carolina.

Let X and Y be completely regular spaces. THEOREM. *If X is first countable, then $X \times Y$ is pseudo-compact if and only if X and Y are both pseudo-compact.* COROLLARY (*using a result of Glicksberg*). *If X is first countable, then $\beta(X \times Y) = \beta X \times \beta Y$ if and only if both X and Y are pseudo-compact.*

19. *Some remarks concerning hyperconvex metrics*, by Professor R. L. Plunkett, Florida State University.

Let X be a compact topological space. It is known that (1) if X has a convex metric with unique segments, then X is an absolute retract (Abstract, *Bull. Amer. Math. Soc.*, vol. 61, 1955, p. 70) and (2) if X has a hyperconvex metric, then X is an absolute retract. (Aronszajn and Panitchpakdi, *Pacific J. Math.*, vol. 6, 1956, pp. 405–439.) Whether (1) implies (2) or vice versa is unknown. In this paper, if d is a metric for X, it is proved that any two of the following statements imply the remaining one: "d is a convex metric with unique segments," "d is a hyperconvex metric," and "X is a dendrite."

20. *Convergence in the overlap topology*, by Professor R. A. Lytle, University of South Carolina.

Suppose (A, T) and (B, T') are topological spaces and f_m is a sequence of evenly continuous functions on A into B. Then f_m converges to the continuous function f in the overlap topology if, and only if, f_m converges to f in the compact open topology.

21. *Retracts from neighborhood retracts*, by Professor W. L. Strother, University of Miami.

In an article to appear in *Proc. Amer. Math. Soc.*, Linda Falcao answers in the affirmative Wojdyslawski's question (*Fund. Math.* 1939) of whether the space of subsets of an absolute retract is again an absolute retract. In this note it is established further that if X is a connected metric absolute neighborhood retract then the space of the subsets of X is an absolute retract.

22. *One+One=One*, by Professor P. R. Halmos, University of Chicago.

An exposition of some of the algebraic and set-theoretic ideas involved in the Banach-Tarski paradox.

23. *Mathematics curriculum survey*, by Professor H. A. Robinson, Agnes Scott College.

From 48 replies to curriculum study suggested at Section Officers Seattle meeting, 21 reported recent changes; 12 appeared to be due to initial sectioning. Two institutions placed lower three-fourths in traditional courses while others studied Griffin's *Introduction to Mathematical Analysis*; 17 offered only algebra and trigonometry to freshmen, 21 analytics and 6 a combination with

calculus; 7 gave integrated courses and 2 universal mathematics; 12 offered intermediate algebra and 2 plane geometry. Sophomore courses were mainly traditional.

24. *Criteria for a logarithmic solution of a certain type of linear differential equation of third order with a regular singular point*, by Professor R. W. Cowan, University of Florida.

A compact expression for the coefficient of a general term in the series solution of the differential equation is obtained by the method of Frobenius. From this expression certain criteria required for a logarithmic solution are developed. These criteria depend upon whether the difference of the roots of the indicial equation are integral multiples of the difference in successive powers of the series solution and the nature of the roots of a certain cubic equation.

25. *On a revised tabulation of fiducial limits for point biserial correlation*, by Professor N. C. Perry, Alabama Polytechnic Institute.

An extensive tabulation of 5% fiducial limits for point biserial correlation (r_{pb}) is incorrect because the supporting non-central t theory of Lev (1949) was partially invalidated in a paper by Tate (1954). The speaker presents the results of a preliminary investigation of the numerical difference between the two theories, obtained by using Tate's normalizing $\tanh^{-1}$ transformation. It was found that fiducial limits agree to two decimals for samples larger than 22, and differ by .01 for samples ranging in size from $n=22$ to $n=11$.

26. *On non-linear behavior of shallow ellipsoidal shells*, by Professor W. A. Nash, University of Florida.

Equilibrium equations are written for an element corresponding to a thin elastic open shell having small curvatures. These equations are specialized to the case of a shell in the form of an elliptic paraboloid. The case of a completely restrained-edge shell subject to uniform normal loading is considered. Infinite series for each of the three orthogonal displacement components are employed and all boundary conditions are satisfied identically. Coefficients in these series are determined and in this manner displacements and stresses throughout the shell are known.

27. *Order relations in the ring of polynomials*, by Professor Trevor Evans, Emory University.

The non-archimedean orderings of $J[x]$, the integral domain of polynomials over the integers, are well-known. It is pointed out in this note that there are uncountably many different ways of introducing an archimedean order into $J[x]$. These are obtained from the isomorphisms of $J[x]$ and sub-domains of the real numbers, x corresponding to different transcendentals.

28. *On a certain limit*, by Mr. Neal Bradley, student, Georgia Institute of Technology.

While trying to find an elementary proof that, if $u(x)$ has a limit as $x\to\infty$, then $\lim_{x\to\infty}[1+u(x)/x]^x=\exp[\lim_{x\to\infty}u(x)]$, it was noticed that the following theorem must hold: *If $u(x)$ is smooth for all sufficiently large x, and $\lim_{x\to\infty}u(x)$ exists, then*:
(1) *Either $u'(x)$ has no limit at all, or $\lim_{x\to\infty}u'(x)=0$.*
(2) *The same is true for $xu'(x)$. (The case $u'(x)\to\pm\infty$ is also excluded.)*

29. *On the summability of series by Nørlund means*, by Visiting Lecturer C. N. Moore, University of South Carolina (Professor Emeritus, University of Cincinnati).

In the case of the well known Cesàro means, the weights used in forming the means are the coefficients in the binomial expansion. A natural extension of this method is to replace the binomial coefficients by the coefficients of an arbitrary power series. Some restriction must be made in order to make the method "regular," that is to make sure that it will sum a convergent series to the proper value. If we represent the coefficients as $c_0, c_1, \cdots$, and set $C_n=\sum_{k=0}^{n}c_k$, the necessary

and sufficient conditions for regularity are $\lim_{n\to\infty} c_{n-m}/C_n=0$ $(0\leqq m\leqq n)$, $\sum_{k=0}^{m}|c_k|<H|C_m|$, when H is a positive constant and $m\geqq 0$.

30. *A development of logarithms using the function concept*, by Professor C. L. Seebeck, Jr., University of Alabama, and Professor John Jewett, University of Georgia, read by Professor Jewett.

The purpose of this paper is to present a definition of the logarithm as a function by means of properties which are simple enough to be understood readily by freshman students. The authors define the logarithm to be a function L which assigns to each positive real number a real number such that (1) $L(a\cdot b)=L(a)+L(b)$, (2) $L(10)=1$, and (3) if $a>1$, then $L(a)>0$. The usual elementary properties of logarithms follow easily from this definition and it can be shown that the function L agrees with the logarithm function defined by other methods.

31. *The mathematics in forward looking high school mathematics programs*, by Professor E. D. Nichols, Florida State University.

Contents of programs which most likely will exert influence on future courses are examined. The program of the University of Illinois Committee on School Mathematics has many interesting features. This group has developed materials which are being used in public schools under typical conditions. These materials represent drastic changes in content. Among aspects examined in this paper are the following: emphasis on distinction between a name of a thing and a thing itself; concept of variable; elementary aspects of set theory; linear and quadratic equations in terms of sets of points, intersection and union of sets.

32. *Mathematics in an industrial computing center*, by Dr. R. A. Willoughby, The Babcock & Wilcox Company, Atomic Energy Division, read by Mr. G. A. Duncan, Lockheed Aircraft Company, Marietta, Georgia.

The application of digital computers to the solution of scientific and technical problems in industry and the numerical analysis associated with the solution of these problems are placing a fresh emphasis on the basic concepts of undergraduate mathematics and their relation to physical problems. A broad horizon of job opportunities are opening to mathematics majors. It may well be that industrial experience will greatly benefit even those mathematics graduates who intend to become high school or college teachers.

33. *The stress distribution in a rotating disk of orthotropic material*, by Professor C. B. Smith, University of Florida.

A circular disk of orthotropic material is considered to be rotating about its center at a constant angular velocity. Since the thickness of the disk is taken to be small as compared with its radius, variation of stresses over thickness can be neglected and the problem is assumed to be a two-dimensional one. Usual equilibrium equations are modified to include body forces given by a potential function. Solution is obtained by aid of complex variables, and shows that stress distribution is much more complicated than that of the corresponding problem for an isotropic disk.

34. *Note on a system of differential equations in reactor statics*, by Professor J. A Nohel, Georgia Institute of Technology.

The differential equations for the fast and slow flux in a two region nuclear reactor form a system of two linear second order differential equations with variable coefficients; moreover, one of these coefficients is unknown. In treatments previously given this coefficient is assumed to be a constant. In this paper the above system is solved in general without this assumption, but subject to the condition that the unknown coefficient is analytic in some region containing the origin. Special cases of practical interest are then considered.

35. *A recurrence matrix for the response of an airplane in landing*, by Professor R. E. Wheeler, Howard College.

The problem of finding the time during landing at which there is a maximum stress at different positions on an airplane wing introduces interesting applications of elementary matrix theory and difference equations. By considering solutions of a difference equation at various positions on the wing, matrix relationships can be obtained which when simplified reduce to a recurrence matrix that gives the displacements at any particular time in terms of three preceding displacements. These can be used to find the time and the amount of the maximum stress.

36. *Nonlinear creep in pin-jointed structures*, by Professor R. C. Meacham, University of Florida.

Determination of the strains and stresses in a pin-jointed truss structure, the material of which obeys an mth power creep rate law, is shown to depend upon solving a system of non-linear, ordinary differential equations. The solution can be effected for any truss structure which contains only one redundant member; for higher degrees of redundancy the problem becomes formidable.

37. *Recursion and interrelations for Miles-Williams biharmonics*, by Professor M. C. Wicht, North Georgia College.

After a brief discussion of basic sets of polyharmonic polynomials, the author develops for the Miles-Williams biharmonics (this MONTHLY, vol. 63, 1956, p. 528) the recursion formula:

$$B_\beta^n = \cos^2(\beta\pi/2)[xB_\beta^{n-1} - (n-\beta-1)yB_{\beta+1}^{n-1}] + \sin^2(\beta\pi/2)[y/nB_{\beta-1}^{n-1} + (n-\beta)xB_\beta^{n-1}],$$

where $\beta = 0, 1, 2, 3$. Further, the author develops a set of eight relations between the Miles-Williams biharmonics and the biharmonics $P_n(x, y)$ of Zweiling.

38. *A nomogram for reducing the cubic equation*, by Professor R. G. Blake, University of Florida.

A design determinant is developed for a nomogram to determine p and q in the transformation $x^3+ax^2+bx+c=0$ to $X^3+pX+q=0$ by the substitution $x=X-a/3$.

39. *Some remarks on functional completeness*, by Professor F. L. Hardy, Emory University.

Consider the finite set of natural numbers $1, \cdots, m$. It is shown that all possible functions from this set to itself can be generated by a single function of two variables, the operation being substitution. This means in m-valued logics that all functionally complete logics may be given by a single truth-table in the same way as the 2-valued logic may be given by the Sheffer stroke function.

40. *On simultaneous reduction of sets of normal real matrices to canonical form*, by Mr. R. D. Boswell, Jr., University of Georgia.

An orthogonal matrix T is said to reduce a normal real matrix A if $T^{-1}AT$ is a block diagonal matrix with entries which are either 1×1 blocks or 2×2 blocks of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

It is proved that if F_t, $-\infty < t < \infty$, is a one-parameter group of $n\times n$ normal real matrices, then there exists a single orthogonal matrix S which reduces F_t for each real t.

41. *A method of reducing a positive ternary quadratic form*, by Professor P. B. Patterson, University of Florida.

It is shown by means of the minimum values of the coefficients that the positive ternary

quadratic form can be transformed into a reduced form by a certain set of three transformations.

42. *A careful study of the equation*, $x=f(x)$, by Professor W. R. Mann, University of North Carolina, and Doctor R. A. Willoughby, The Babcock and Wilcox Company, Atomic Energy Division, read by Professor Mann.

The problem of solving $x=1+\sin x$ accurate to a specified number of significant digits is used as a motivation for a careful investigation of a number of important topics in advanced calculus and numerical analysis (e.g. mean value theorem, convergence of sequences, Taylor series, numerical uncertainty and error analysis). In studying the equation $x=f(x)$ strong intuitive appeal can be given through the use of analytic geometry. Properly guided, the student can have the thrill of discovering many interesting and important properties.

43. *A lemma in matrices*, by Mr. R. T. Stubbs, graduate student, Georgia Institute of Technology.

Let A be any $n\times n$ complex constant matrix with characteristic roots λ_i, $i=1, \cdots, n$, such that $\Re(\lambda_i) \leqq -\mu<0$, $i=1, \cdots, n$. If σ is a constant such that $0<\sigma<\mu$, then there exists a constant K greater than zero such that $\|e^{tA}\| \leqq Ke^{-\sigma t}$, $t\geqq 0$ where K depends only on μ, σ, and A; and $\|e^{tA}\|$ is defined to be the sum of the absolute values of the elements of the matrix e^{tA}. Although the above result is well known, it is not readily found in the literature.

H. A. ROBINSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, State Teachers College, Mankato, October 5, 1957.
MISSOURI
NEBRASKA
NEW JERSEY, Fairleigh Dickinson University, Rutherford, November 2, 1957.
NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 30, 1957.
NORTHERN CALIFORNIA, San Francisco State College, January, 1958.
OHIO
OKLAHOMA, Oklahoma City University, October 25, 1957.
PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.
PHILADELPHIA, November 30, 1957.
ROCKY MOUNTAIN
SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.
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